

Yang-Mills theory and Tamagawa Numbers:

the fascination of unexpected links in mathematics

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February 3, 2008

1 Introduction and overview

This article is an expanded version of the third author's presidential address to the London Mathematical Society in November 2005, which was based on work with the other two authors over the preceding year. The title was inspired by Michael Atiyah's presidential address delivered in 1976 [Ati78] in which he said

“The aspect of mathematics which fascinates me most is the rich interaction between its different branches, the unexpected links, the surprises.”

The unexpected link which is the topic of this article was remarked on by Atiyah himself and his collaborator Raoul Bott in their fundamental 1983 paper [AB83] on the Yang-Mills equations over Riemann surfaces. In this paper Atiyah and Bott used ideas coming from Yang-Mills theory and equivariant Morse theory to derive inductive formulae for the Betti numbers of the moduli spaces $\mathcal{M}(n, d)$ of stable vector bundles of rank n and degree d over a fixed compact Riemann surface C of genus $g \geq 2$, when n and d are coprime. (We will assume throughout this introduction that n and d are coprime integers with $n > 0$.) Equivalent formulae had been obtained earlier by Harder and Narasimhan [HN75] and Desale and Ramanan [DR75] using arithmetic techniques and the Weil conjectures. In the latter approach a crucial ingredient was the fact, proved by Weil, that the volume of a certain locally symmetric space attached to SL_n with respect to a canonical measure – an invariant known as the Tamagawa number of SL_n – is 1.

Atiyah and Bott observed that although the two methods came from very different branches of mathematics, namely arithmetic and physics, there was a formal correspondence between them, with the Tamagawa number of SL_n (or equivalently the function field analogue of the Siegel formula) playing, roughly speaking, the rôle of the cohomology of the classifying space of the gauge group in the Atiyah-Bott approach. They asked for a deeper understanding of this observation and in particular for a geometric explanation, exploiting the analogy with equivariant cohomology, of the fact that the Tamagawa number of SL_n is 1. Contributions since then towards such understanding have included work by Bifet, Ghione and Letizia [Bif89, BGL94], providing another inductive procedure

for calculating the Betti numbers of the moduli spaces which is in some sense intermediate between the arithmetic approach and the Yang-Mills approach, and more recently, work by Teleman, Behrend, Dhillon and others on the moduli stack of bundles over C [BDa, BDb, Dhi06, Tel98].

The formal correspondence observed by Atiyah and Bott between the inductive formulae obtained from the Yang-Mills and arithmetic points of view arises because both depend on stratifications of spaces whose points represent vector bundles over C , with the stratification induced by the “Harder-Narasimhan types” of the bundles. In each case it is the open stratum, corresponding to semistable bundles, which needs to be understood, and the inductive calculation comes from combining a simple description of the other strata (in terms of semistable strata for the corresponding problem with strictly smaller values of n and varying values of d) together with knowledge of the whole space. In the arithmetic approach the space to be stratified is a coset space of the group $SL_n(\mathbb{A}_K)$ associated with the adèle ring \mathbb{A}_K of the function field K of a nonsingular projective curve over a finite field. The Tamagawa measure of the whole space is the (countably infinite) sum of the measures of the strata. In the Yang-Mills approach Atiyah and Bott stratify the infinite-dimensional affine space \mathcal{A} of unitary connections on a fixed C^∞ bundle of rank n and degree d over C , and they show that the stratification is “equivariantly perfect” with respect to the gauge group, so that the equivariant cohomology algebra of \mathcal{A} is isomorphic as a vector space to the sum over the strata of their equivariant cohomology algebras, shifted in degree by their real codimensions.

The cohomology of the moduli spaces $\mathcal{M}(n, d)$ of bundles over a compact Riemann surface has, of course, more structure than is revealed by its Betti numbers. It has been an object of study for several decades, first when $n = 2$ and then for general n , from many points of view including mathematical physics [AB83, Wit92], matrix divisors [BGL94, Bif89], extended moduli spaces [JW94, HJ94, Jef94], group valued moment maps and quasi-Hamiltonian reductions [AMM98, AMW00, AMW01, AMW02, MW99, Mei05] and others [Dhi06, Kir86a, Kir86b, JK98, JKKW06], though the ideas of Atiyah and Bott have been fundamental for much of this progress.

Following a long line of work initiated by Grothendieck in the 1960s, Bloch and Voevodsky have developed sophisticated versions of cohomology (motivic cohomology, see [Blo86, Voe00, MVW06]) for algebraic varieties defined over arbitrary fields. Voevodsky and Morel had a broader vision (see [MV99]): they sought to build a homotopy theory for schemes over a field k where the affine line plays a role analogous to that played by the unit interval in ordinary homotopy theory. In the resulting homotopy category, called the \mathbb{A}^1 or motivic homotopy category, there are two different analogues of the circle: the simplicial circle which is obtained from the affine line by glueing together 0 and 1 (represented by the affine nodal cubic curve) and the Tate circle $\mathbb{A}^1 - \{0\}$. Corresponding to these two analogues of the circle, a pair of integers index the motivic cohomology groups of a smooth scheme X over a field k , reflecting the fact that these groups form a bigraded ring. Motivic cohomology groups for smooth schemes over k have many properties analogous to ordinary singular cohomology, including an

appropriate form of homotopy invariance, Mayer-Vietoris and Gysin long exact sequences. However, motivic cohomology encodes far more information; for example, in contrast with ordinary singular cohomology, the motivic cohomology of a point $\mathrm{Spec} k$ is quite large, with its degree (p, p) part being isomorphic to the p -th Milnor K -group of the field k (see [MVW06] Chapter 4). Furthermore, if X is a smooth k -scheme, the motivic cohomology groups $H^{2p,p}(X, \mathbb{Z})$ are isomorphic to the Chow groups of codimension p cycles on X modulo rational equivalence.

The aim of this article is threefold: to announce results, produced in the setting of \mathbb{A}^1 -homotopy theory, on the motivic cohomology (and hence its many realizations) of quotients in the sense of Mumford's geometric invariant theory or GIT [ADK], generalizing what was known for singular cohomology of GIT quotients; to explain how methods used over the last three decades to study the singular cohomology of the moduli spaces $\mathcal{M}(n, d)$ can thence be adapted to study the motivic cohomology of moduli spaces of bundles on a smooth projective curve C over an algebraically closed field k ; and at the same time to extend our understanding of the link between Yang-Mills theory and Tamagawa numbers remarked on by Atiyah and Bott, by re-examining some of their essentially homotopy theoretic considerations in a more algebraic modern light.

This study is based on an adaptation to the setting of motivic cohomology of the inductive methods obtained in the third author's thesis [Kir84] for calculating the Betti numbers of a GIT quotient [MFK94] of a nonsingular complex projective variety X by a linear action of a complex reductive group G . These methods were themselves inspired by the work of Atiyah and Bott [AB83] and involve applying equivariant Morse theory to the norm-square of an appropriate moment map, which is the analogue of the Yang-Mills functional in [AB83]. The associated stratification of X has an alternative purely algebraic description, independent of Morse theory, which is valid much more generally than just over the complex numbers \mathbb{C} : the semistable points of X (in the sense of GIT) form an open stratum, and the other strata can be described inductively in terms of the semistable points of nonsingular projective subvarieties of X under appropriately linearized actions of reductive subgroups of G . In the finite-dimensional algebro-geometric setting, most of the results of [Kir84] can be adapted to apply to motivic cohomology [ADK]. These results can then be used to study the motivic cohomology of moduli spaces of vector bundles over a curve C .

Much beautiful work has been done recently on the moduli of bundles over a curve which can be used to extract similar information [dB01, dB02, BDa, Dhi06]. Our aim here is to derive this information as an application of a general theory of cohomology of GIT quotients, and to offer some insights gleaned from returning to a mapping space perspective on the moduli space of bundles.

Outline

In §2 of this paper we describe the two equivalent inductive procedures for calculating the Betti numbers of $\mathcal{M}(n, d)$ provided by the arithmetic and Yang-Mills approaches. The Yang-Mills approach of Atiyah and Bott [AB83], which

uses equivariant Morse theory, has as its basic ingredient a simple description given in [AB83] of the cohomology of the classifying space of the gauge group, while the arithmetic approach of [HN75, DR75] uses Tamagawa measures and reduces to the function field version of the Siegel formula, or equivalently to the fact that the Tamagawa number of SL_n is 1.

These two approaches rely on what can be regarded as infinite-dimensional quotient constructions of the moduli spaces $\mathcal{M}(n, d)$. A third approach, closely related to that used by Bifet, Ghione and Letizia in [BGL94], involves the construction of the moduli spaces $\mathcal{M}(n, d)$ as finite-dimensional GIT quotients which can be regarded as finite-dimensional approximations to the Yang-Mills construction. This involves studying spaces of maps into Grassmannians and matrix divisors, and provides an inductive calculation of the Betti numbers of the moduli spaces in terms of the cohomology of symmetric powers of the curve C (this cohomology is well known [Mac62]). Here the inductive formulae can be obtained from equivariant Morse theory as in the Yang-Mills approach, but in this finite-dimensional setting the Weil conjectures provide an alternative derivation by using the algebraic description of the Morse strata and counting points on associated varieties defined over finite fields.

In §3 we move into the motivic world; our aim is to show that the finite dimensional GIT construction of $\mathcal{M}(n, d)$ used in the third approach to calculate Betti numbers can also be used to study the motivic cohomology of $\mathcal{M}(n, d)$. The first step is to make sense of equivariant motivic cohomology; this is done using a straightforward modification of the Borel construction in topology using work of Morel and Voevodsky. In §4 we describe how to adapt the methods of [Kir84] on the cohomology of GIT quotients to the setting of motivic cohomology, and in §5 we apply these to obtain an inductive description of the motivic cohomology groups of $\mathcal{M}(n, d)$ in terms of the motivic cohomology of products of symmetric powers of the curve C .

In the final section §6 we leave motivic cohomology and return to the original question of the relationship between Yang-Mills theory and Tamagawa measures. We observe that the inductive procedure for calculating the Betti numbers of $\mathcal{M}(n, d)$ which uses finite-dimensional approximations to the Yang-Mills picture, via maps into Grassmannians and matrix divisors, is directly equivalent to the Yang-Mills approach through a generalization of Segal's theorem on the topology of spaces of rational functions [Seg79]. This theorem makes precise the sense in which the Yang-Mills approach is an infinite-dimensional limit of the corresponding procedure using maps into Grassmannians. In a similar way the arithmetic methods used by Harder and Narasimhan to provide an inductive calculation of the Betti numbers of $\mathcal{M}(n, d)$ can be regarded as an infinite-dimensional limit of the alternative finite-dimensional approach which involves counting points on associated varieties over finite fields. The Weil conjectures then provide the final link in the chain connecting the Yang-Mills and arithmetic viewpoints.

Acknowledgements

The authors would like to thank Tamas Hausel, Michael McQuillan, James Parson, Burt Totaro and Akshay Venkatesh for helpful discussions and comments.

2 A review of the classical constructions

The moduli spaces of stable vector bundles on a smooth projective curve C over a field k have different incarnations depending on the field k . Over a field k with sufficiently many elements, these moduli spaces can be constructed by means of geometric invariant theory. If $k = \mathbb{C}$ and we view the complex points of C as a compact Riemann surface, we have a differential geometric construction using an interpretation of stable vector bundles in terms of connections. If $k = \mathbb{F}_q$, there is an interpretation of such vector bundles in terms of *adèles*. We review the last two of these constructions in this section. We will discuss the GIT construction in §5.

Historically, the computation of the cohomology of these moduli spaces was first achieved using number theory and then using Yang-Mills theory, while algebraic geometry provides a logical bridge between these two contexts. In this section we will review the differential geometric and the arithmetic approaches to computing the cohomology of the moduli spaces. Throughout this section, C will denote a compact Riemann surface or a smooth projective algebraic curve defined over a field k . We abuse notation in this way to emphasize the interchangeability of the different points of view, the object under consideration being clear from its context.

2.1 Yang-Mills and Riemann surfaces

Let C be a compact Riemann surface of genus g . In order to avoid having to consider special cases, we will assume throughout that g is at least 2. By convention, holomorphic vector bundles on C will be denoted by calligraphic letters $\mathcal{E}, \mathcal{F}, \dots$ and the underlying C^∞ -complex vector bundles will be denoted by Roman letters E, F, \dots . Furthermore, all bundles will be assumed to be C^∞ -bundles; thus the term complex vector bundle should be read as C^∞ -complex vector bundle. Given a complex vector bundle E , the bundle of frames of E is a $GL_n(\mathbb{C})$ -principal bundle. This construction defines a bijection between the set of complex vector bundles of rank n on C and the set of $GL_n(\mathbb{C})$ -principal bundles (with inverse given by forming the vector bundle associated with the standard n -dimensional representation of GL_n). By abuse of terminology, we will use the same notation E for the $GL_n(\mathbb{C})$ -principal bundle associated with a complex vector bundle E .

Topologically, complex vector bundles of fixed rank n on C are classified by homotopy classes of maps to the space BGL_n , which is (homotopy equivalent to) the Grassmannian of n -dimensional quotients of an infinite dimensional complex vector space. The degree or first Chern class $d \in H^2(C, \mathbb{Z})$, which can be identified with \mathbb{Z} by pairing with the fundamental class, is also a topological

invariant and the rational number $\mu(E) = d/n$ is called the *slope* of E . We will also write $\mu(\mathcal{E})$ for the slope of the complex vector bundle E underlying a holomorphic vector bundle \mathcal{E} .

If \mathcal{E} is a holomorphic vector bundle, we call \mathcal{E} *stable* (respectively *semistable*) if every proper holomorphic subbundle \mathcal{F} of \mathcal{E} satisfies $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$). Note that when n and d are coprime, every semistable bundle of rank n and degree d is stable.

Let E be a fixed complex vector bundle of rank n and degree d over C . The group $\text{Aut}_C(E)$ of all complex vector bundle automorphisms of E is called the *complexified gauge group* of E in the Yang-Mills context (see below) and denoted \mathcal{G}_C . Let $\mathcal{C} = \mathcal{C}(n, d)$ denote the space of all holomorphic structures on E , and let \mathcal{C}^s (respectively \mathcal{C}^{ss}) be the subset of \mathcal{C} consisting of stable (respectively semistable) holomorphic structures on E . Since C is one complex-dimensional, all almost-complex structures on E are automatically integrable, and holomorphic structures on E are specified by elements of an infinite-dimensional complex affine space whose vector space of translations is isomorphic to the space $\Omega^{0,1}(C, \text{End}(E))$. The complexified gauge group \mathcal{G}_C acts on \mathcal{C} by bundle automorphisms, and isomorphism classes of holomorphic vector bundles on C are in bijection with \mathcal{G}_C -orbits in \mathcal{C} . In modern terms, we can identify the *quotient stack* $\mathcal{C}/\mathcal{G}_C$ with the *moduli stack* of vector bundles on C , but to obtain a well behaved moduli *space* we restrict ourselves to stable bundles.

Let $\overline{\mathcal{G}}_C$ denote the quotient of \mathcal{G}_C by its central subgroup, isomorphic to \mathbb{C}^* , corresponding to scalar multiples of the identity automorphism of E . This quotient group $\overline{\mathcal{G}}_C$ acts freely on \mathcal{C}^s (which equals \mathcal{C}^{ss} when n and d are coprime) and the quotient space $\mathcal{C}^s/\overline{\mathcal{G}}_C$ can be naturally identified with $\mathcal{M}(n, d)$, the moduli space of stable holomorphic bundles on C of rank n and degree d .

Atiyah and Bott identify the space \mathcal{C} of holomorphic structures on E with a space of unitary connections on E in order to apply Yang-Mills theory to study $\mathcal{M}(n, d)$. A $U(n)$ -principal bundle on C will be called a *unitary bundle*; unitary bundles on C are classified by homotopy classes of maps from C to the classifying space $BU(n)$. The set of isomorphism classes of unitary bundles and the set of isomorphism classes of $GL_n(\mathbb{C})$ -principal bundles are in bijection as the inclusion $U(n) \hookrightarrow GL_n(\mathbb{C})$ is a homotopy equivalence and hence induces a homotopy equivalence $BU(n) \longrightarrow BGL_n$. Thus, a $GL_n(\mathbb{C})$ -bundle E admits the structure of a unitary bundle and all such structures are equivalent up to $GL_n(\mathbb{C})$ -bundle automorphisms. A Hermitian structure on a complex vector bundle E is a choice of unitary bundle structure underlying the given $GL_n(\mathbb{C})$ -bundle structure; let us fix such a Hermitian structure on E . The gauge group \mathcal{G} is then the group of unitary bundle automorphisms of E ; it is homotopy equivalent to its complexification which is the group of complex automorphisms of E we have already denoted by \mathcal{G}_C . The automorphism group of any unitary bundle contains a central subgroup isomorphic to $U(1)$. Let $\overline{\mathcal{G}}$ denote the quotient of \mathcal{G} by this central subgroup; then $\overline{\mathcal{G}}_C$ is the complexification of $\overline{\mathcal{G}}$.

There is a canonical affine linear isomorphism of the space \mathcal{A} of unitary connections on the complex vector bundle E with the space \mathcal{C} of holomorphic structures on E ([AB83] p. 570). The Yang-Mills functional on \mathcal{A} associates

with any unitary connection A on E the integral over C of the norm square of its curvature F_A . Atiyah and Bott apply the ideas of equivariant Morse theory to the Yang-Mills functional on \mathcal{A} (or equivalently on \mathcal{C}).

Recall that the equivariant cohomology $H_G^*(Y)$ of a topological space Y on which a group G acts can be defined as the ordinary cohomology of the Borel quotient $EG \times_G Y$ where $EG \rightarrow BG$ is the universal G -bundle over the classifying space BG of G . When G acts freely on Y the natural map $EG \times_G Y \rightarrow Y/G$ has contractible fibres and induces an isomorphism $H^*(Y/G) \cong H_G^*(Y)$, while when Y is contractible the map $EG \times_G Y \rightarrow BG$ induces an isomorphism between $H_G^*(Y)$ and $H^*(BG)$. Since \mathcal{C} is an affine space it is contractible, and so

$$H_{\mathcal{G}_{\mathbb{C}}}^*(\mathcal{C}) \cong H^*(B\mathcal{G}_{\mathbb{C}}).$$

This algebra is easy to describe explicitly: over \mathbb{Q} it is freely generated as a polynomial algebra tensored with an exterior algebra by the Künneth components of the equivariant Chern classes of the universal bundle on $\mathcal{C} \times X$ and has Poincaré series

$$\begin{aligned} P_t(B\mathcal{G}_{\mathbb{C}}) &\stackrel{\text{def}}{=} \sum_{i \geq 0} t^i \dim_{\mathbb{Q}} H^i(B\mathcal{G}_{\mathbb{C}}, \mathbb{Q}) \\ &= \frac{\prod_{j=1}^n (1 + t^{2j-1})^{2g}}{(1 - t^{2n}) \prod_{j=1}^{n-1} (1 - t^{2j})^2}. \end{aligned} \tag{1}$$

Since $\overline{\mathcal{G}_{\mathbb{C}}}$ acts freely on \mathcal{C}^s , the identification of smooth manifolds $\mathcal{M}(n, d) \cong \mathcal{C}^s / \mathcal{G}_{\mathbb{C}} \cong \mathcal{C}^s / \overline{\mathcal{G}_{\mathbb{C}}}$ induces the following isomorphisms on cohomology:

$$H^*(\mathcal{M}(n, d)) \cong H^*(\mathcal{C}^s / \overline{\mathcal{G}_{\mathbb{C}}}) \cong H_{\mathcal{G}_{\mathbb{C}}}^*(\mathcal{C}^s)$$

and

$$H_{\mathcal{G}_{\mathbb{C}}}^*(\mathcal{C}^s) \cong H^*(BC^*) \otimes H_{\mathcal{G}_{\mathbb{C}}}^*(\mathcal{C}^s).$$

Atiyah and Bott study the cohomology of the moduli space $\mathcal{M}(n, d)$ when n and d are coprime (so that $\mathcal{C}^s = \mathcal{C}^{ss}$) by showing that the restriction map $H_{\mathcal{G}_{\mathbb{C}}}^*(\mathcal{C}) \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^*(\mathcal{C}^{ss})$ is surjective. In order to do this, they consider the Yang-Mills (or Atiyah-Bott-Shatz) stratification of \mathcal{C} ; this is a stratification of \mathcal{C} by $\overline{\mathcal{G}_{\mathbb{C}}}$ -stable submanifolds, which is the ‘Morse stratification’ induced by the Yang-Mills functional but also has a purely algebraic description as follows.

All holomorphic vector bundles on C are algebraic. If C is an algebraic curve over any field k , and \mathcal{E} is an algebraic vector bundle over C , then \mathcal{E} has a canonical *Harder-Narasimhan filtration*. This is an increasing filtration by algebraic subbundles

$$0 = F_0(\mathcal{E}) \subset F_1(\mathcal{E}) \subset \cdots \subset F_r(\mathcal{E}) = \mathcal{E} \tag{2}$$

uniquely determined by the conditions that $F_j(\mathcal{E})/F_{j-1}(\mathcal{E}) = gr_j(\mathcal{E})$ is semistable for $1 \leq j \leq r$, of degree d_j and rank n_j , say, and that the sequence of slopes $\mu(gr_j(\mathcal{E})) = d_j/n_j$ satisfies

$$\mu(gr_1(\mathcal{E})) > \mu(gr_2(\mathcal{E})) > \cdots > \mu(gr_r(\mathcal{E})).$$

The Harder-Narasimhan type of E is defined to be the decreasing sequence $\mu = (\mu_1, \dots, \mu_n)$ of rational numbers in which $\mu(gr_j(\mathcal{E})) = d_j/n_j$ appears n_j times. The Yang-Mills stratum \mathcal{C}_μ is then the subset of \mathcal{C} consisting of all holomorphic structures of Harder-Narasimhan type μ on E .

Since the Harder-Narasimhan filtration of a holomorphic vector bundle is canonically defined, the assignments $\mathcal{E} \mapsto gr_j \mathcal{E}$ induce a map from the space of vector bundles with a fixed Harder-Narasimhan type $(d_1/n_1, \dots, d_r/n_r)$ as above to the product of the spaces of bundles of degree d_j and rank n_j . If we let $\mathcal{G}_{\mathbb{C}}(n_j, d_j)$ denote the complexified gauge group of the smooth vector bundle underlying $gr_j \mathcal{E}$, the map just defined (together with the Künneth formula) gives an isomorphism:

$$H_{\mathcal{G}_{\mathbb{C}}}^*(\mathcal{C}_\mu) \cong \bigotimes_{j=1}^r H_{\mathcal{G}_{\mathbb{C}}(n_j, d_j)}^*(\mathcal{C}^{ss}(n_j, d_j)). \quad (3)$$

The unique open stratum of the Yang-Mills stratification is \mathcal{C}^{ss} , and when n and d are coprime we have

$$H_{\mathcal{G}_{\mathbb{C}}}^*(\mathcal{C}^{ss}) \cong H^*(B\mathbb{C}^*) \otimes H^*(\mathcal{M}(n, d)).$$

The complexified gauge group $\mathcal{G}_{\mathbb{C}}$ acts on $\mathcal{C} = \mathcal{C}(n, d)$ preserving the Yang-Mills stratification. The Yang-Mills strata may be totally ordered so that the closure of a stratum \mathcal{C}_μ is contained in the union of lower-dimensional strata $\mathcal{C}_{\mu'}$ with $\mu' \geq \mu$. We let U_μ denote the open subset of \mathcal{C} obtained by taking the union of all strata $\mathcal{C}_{\mu'}$ for $\mu' \leq \mu$. We can then consider the inclusion of \mathcal{C}_μ into U_μ and the associated Thom-Gysin sequence:

$$\dots \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^{j-2c_\mu}(\mathcal{C}_\mu) \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^j(U_\mu) \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^j(U_\mu - \mathcal{C}_\mu) \rightarrow \dots$$

where c_μ is the codimension of the complement of U_μ in $\mathcal{C}_{n,d}$. The Yang-Mills stratification is equivariantly perfect in the sense that these Thom-Gysin sequences break up into short exact sequences

$$0 \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^{j-2c_\mu}(\mathcal{C}_\mu) \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^j(U_\mu) \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^j(U_\mu - \mathcal{C}_\mu) \rightarrow 0.$$

The integer c_μ can be computed in terms of ranks and degrees appearing in the Harder-Narasimhan type $\mu = (d_1/n_1, \dots, d_r/n_r)$:

$$c_\mu = \sum_{\ell > j} (n_\ell d_j - n_j d_\ell + n_\ell n_j (g-1)). \quad (4)$$

Atiyah and Bott show that the Yang-Mills stratification is equivariantly perfect by considering the composition of the Thom-Gysin map $H_{\mathcal{G}_{\mathbb{C}}}^{j-2c_\mu}(\mathcal{C}_\mu) \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^j(U_\mu)$ with restriction to \mathcal{C}_μ , which is multiplication by the equivariant Euler class e_μ of the normal bundle to \mathcal{C}_μ in \mathcal{C} . They show that e_μ is not a zero-divisor in $H_{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}_\mu)$ and deduce that the Thom-Gysin maps $H_{\mathcal{G}_{\mathbb{C}}}^{j-2c_\mu}(\mathcal{C}_\mu) \rightarrow H_{\mathcal{G}_{\mathbb{C}}}^j(U_\mu)$ are all injective.

Choosing splittings of the short exact sequences above then gives the following direct sum decomposition (of rational vector spaces):

$$H_{\mathcal{G}_{\mathbb{C}}}^j(\mathcal{C}, \mathbb{Q}) \cong \bigoplus_i H_{\mathcal{G}_{\mathbb{C}}}^{j-2c_{\mu}}(\mathcal{C}_{\mu}, \mathbb{Q}). \quad (5)$$

One may derive the inductive formulas obtained by Atiyah and Bott in [AB83] for the equivariant Betti numbers of $\mathcal{C}_{n,d}^{ss}$ (and hence when n and d are coprime for the Betti numbers of $\mathcal{M}(n,d)$) by combining equations (3), (1) and (5). In terms of the equivariant Poincaré series

$$P_t^{\mathcal{G}_{\mathbb{C}}}(X) \stackrel{def}{=} \sum_{i \geq 0} t^i \dim_{\mathbb{Q}} H_{\mathcal{G}_{\mathbb{C}}}^i(X, \mathbb{Q})$$

they are given by

$$P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}^{ss}) = P_t(B\mathcal{G}_{\mathbb{C}}) - \sum_{\mu \neq (d/n, \dots, d/n)} P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}_{\mu}) \quad (6)$$

where $P_t(B\mathcal{G}_{\mathbb{C}})$ is given by (1) and if $\mu = (d_1/n_1, \dots, d_r/n_r)$ then by (3) we have

$$P_t^{\mathcal{G}_{\mathbb{C}}}(\mathcal{C}_{\mu}) = \prod_{j=1}^r P_t^{\mathcal{G}_{\mathbb{C}}(n_j, d_j)}(\mathcal{C}^{ss}(n_j, d_j)).$$

The case $n = 2$.

Every line bundle over C is stable, and the moduli spaces $\mathcal{M}(1,d)$ are topologically tori $\mathbb{C}^g/\mathbb{Z}^{2g}$; we have

$$P_t(\mathcal{M}(1,d)) = P_t(B\overline{\mathcal{G}}_{\mathbb{C}}(1,d)) = (1+t)^{2g}.$$

When $n = 2$ and d is odd, the Harder-Narasimhan filtration of an unstable bundle \mathcal{E} is very simple: it is given by

$$0 \subset \mathcal{L} \subset \mathcal{E}$$

where \mathcal{L} is a line subbundle of \mathcal{E} of degree $d_1 > d/2$. Thus the inductive formula (6) takes the form

$$\begin{aligned} P_t^{\mathcal{G}_{\mathbb{C}}(2,d)}(\mathcal{C}^{ss}(2,d)) &= P_t(B\mathcal{G}_{\mathbb{C}}(2,d)) \\ &- \sum_{d_1 > d/2} t^{2(2d_1-d+g-1)} P_t^{\mathcal{G}_{\mathbb{C}}(1,d_1)}(\mathcal{C}^{ss}(1,d_1)) P_t^{\mathcal{G}_{\mathbb{C}}(1,d-d_1)}(\mathcal{C}^{ss}(1,d-d_1)) \\ &= \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^4)(1-t^2)^2} - \sum_{j=0}^{\infty} t^{2(2j+g)} \left(\frac{(1+t)^{2g}}{(1-t^2)} \right)^2. \end{aligned} \quad (7)$$

It follows that when d is odd

$$\begin{aligned} P_t(\mathcal{M}(2, d)) &= (1 - t^2)P_t^{\mathcal{G}_C(2, d)}(\mathcal{C}^{ss}(2, d)) = \frac{(1 + t)^{2g}}{(1 - t^2)} \left(\frac{(1 + t^3)^{2g}}{(1 - t^4)} - \frac{t^{2g}(1 + t)^{2g}}{(1 - t^4)} \right) \\ &= \frac{(1 + t)^{2g}((1 + t^3)^{2g} - t^{2g}(1 + t)^{2g})}{(1 - t^2)^2(1 + t^2)} \end{aligned} \quad (8)$$

which is a polynomial of degree $6g - 6$ in t if $g \geq 2$.

2.2 Tamagawa numbers and curves over finite fields

The Weil conjectures

The computation of the Betti numbers of the moduli spaces $\mathcal{M}(n, d)$ of bundles of coprime rank n and degree d over a nonsingular complex projective curve C by Atiyah and Bott was preceded by equivalent inductive formulae presented by Harder, Narasimhan, Desale and Ramanan [HN75, DR75]. These computations utilize the Weil conjectures, proved by Deligne, and a computation of the Tamagawa number for SL_n , perhaps originally due to Siegel, but appearing explicitly in work of Weil [Wei82]. Let us briefly review the setting of the Weil conjectures as it serves to motivate “motivic” ideas.

Let \mathbb{F}_q be a finite field with q elements, where q is a power of a prime p . Suppose that X is a finite type scheme over \mathbb{F}_q . Following Weil, one defines the zeta function of X as the following formal power series with rational coefficients:

$$Z_X(t) = \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})| \frac{t^r}{r}\right).$$

Thus $Z_X(t)$ is a generating function for the numbers $|X(\mathbb{F}_{q^r})|$ of \mathbb{F}_{q^r} -rational points of X .

Suppose now that X is a smooth projective variety over \mathbb{F}_q of dimension n . Weil conjectured that $Z_X(t)$ has the following three properties, the second and third of which are analogous to properties of the Riemann zeta function. First, $Z_X(t)$ is a rational function of t , i.e. it is a quotient of polynomials with rational coefficients. Second, if E is the self-intersection number of the diagonal Δ of $X \times X$, then $Z_X(t)$ satisfies a functional equation:

$$Z_X\left(\frac{1}{q^n t}\right) = \pm q^{\frac{nE}{2}} t^E Z_X(t).$$

Thirdly, $Z_X(t)$ satisfies an analogue of the Riemann hypothesis. More precisely, it is possible to write

$$Z_X(t) = \frac{P_1(t)P_3(t) \cdots P_{2n-1}(t)}{P_0(t)P_2(t) \cdots P_{2n}(t)}$$

where $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$ and all the other $P_i(t)$ are polynomials with integer coefficients that can be written

$$P_i(t) = \prod (1 - \alpha_{ij}t)$$

with α_{ij} some collection of algebraic integers of norm $q^{i/2}$. (The closer analogue of the Riemann zeta function is $\zeta_X(s) = Z_X(q^{-s})$, whose zeros and poles are on the lines $\operatorname{Re} s = j/2$ with $j = 0, 1, \dots, 2n$.) The first two statements of the Weil conjectures were verified (for arbitrary n) first by Dwork and later by Artin and Grothendieck using Grothendieck's theory of étale cohomology. The Riemann hypothesis was later established by Deligne [Del80].

Let $\overline{\mathbb{F}_q}$ denote an algebraic closure of \mathbb{F}_q . Given a finite type scheme X over \mathbb{F}_q , let \bar{X} be the variety over $\overline{\mathbb{F}_q}$ obtained by base change. Let Fr_q denote the geometric Frobenius automorphism of \bar{X} induced by the automorphism $\alpha \mapsto \alpha^q$ of $\overline{\mathbb{F}_q}$. Suppose that ℓ is a prime number not equal to p , and let \mathbb{Q}_ℓ denote the field of ℓ -adic numbers. Then Artin and Grothendieck defined étale cohomology groups $H_{\text{ét}}^i(\bar{X}, \mathbb{Q}_\ell)$ satisfying many properties analogous to usual cohomology groups: these groups are only non-vanishing when $0 \leq i \leq 2n$, satisfy Poincaré duality and have the usual exact sequences (Mayer-Vietoris, Thom-Gysin, etc.). Furthermore, they showed using an ℓ -adic version of the Lefschetz fixed point theorem that $P_i(t)$ could be interpreted as the characteristic polynomial of Fr_q acting on the ℓ -adic étale cohomology group $H_{\text{ét}}^i(\bar{X}, \mathbb{Q}_\ell)$. Thus the degrees of the polynomials $P_i(t)$ are in fact the ranks of these ℓ -adic cohomology groups.

Let $L \hookrightarrow \mathbb{C}$ be an algebraic number field. If X is a smooth projective variety defined over L , then one can use the Weil conjectures to determine the ordinary Betti numbers of X thought of as a complex variety. The essential point is that such an X can be treated via base change both as a variety over a finite field \mathbb{F}_q , for appropriate q , and also (given a choice of embedding $L \hookrightarrow \mathbb{C}$) as a variety over \mathbb{C} , and furthermore the ℓ -adic Betti numbers of the former agree with the ordinary Betti numbers of the latter. Note that this is very much an observation about Betti numbers rather than more refined topological invariants. Indeed, Serre has produced in [Ser64] examples of smooth projective varieties X over L such that simply choosing two different embeddings of L into \mathbb{C} yields two smooth complex varieties which are not homeomorphic to each other, although by the Weil conjectures they must have the same Betti numbers (more completely, it is known that all such “conjugate” complex varieties must have the same “étale homotopy type”).

Let us be a little more precise. Suppose that L is an algebraic number field equipped with a fixed embedding $\phi : L \hookrightarrow \mathbb{C}$ and let \mathcal{O}_L denote the ring of integers in L ; we write \mathfrak{p} for a non-zero prime ideal (necessarily maximal) of \mathcal{O}_L and the quotient $\mathcal{O}_L/\mathfrak{p}$ is a finite field \mathbb{F}_q (where q is a power of some prime p). Suppose that X is a smooth projective variety defined over L ; we will now consider the \mathbb{F}_q -variety that is the “reduction modulo \mathfrak{p} ” of X . Choose a projective embedding of X . Upon clearing denominators, which involves only finitely many prime ideals, we obtain a non-empty open subscheme B' of $\operatorname{Spec} \mathcal{O}_L$ and an associated morphism of schemes $\mathfrak{X}' \rightarrow B'$ which is proper

(indeed projective) over B' and restricts to X over the generic point $\text{Spec } L$ of B' . Since the morphism is generically smooth, shrinking B' if necessary we obtain a non-empty open subscheme B of $\text{Spec } \mathcal{O}_L$ and associated morphism $\mathfrak{X} \rightarrow B$ which is both smooth and projective and which restricts to X over the generic point $\text{Spec } L$. Given a prime ideal \mathfrak{p} whose inclusion in \mathcal{O}_L induces a morphism $\text{Spec } \mathcal{O}_L/\mathfrak{p} \cong \text{Spec } \mathbb{F}_q \rightarrow B$, the image is necessarily a closed point of B . The scheme-theoretic fibre $\mathfrak{X}_{\mathfrak{p}}$ of $\mathfrak{X} \rightarrow B$ over this closed point is the “reduction modulo \mathfrak{p} ” of \mathfrak{X} . Being a fibre product with $\text{Spec } \mathbb{F}_q$, it is necessarily an \mathbb{F}_q -scheme, and by the construction of B it is both smooth and projective. Then the degree of the polynomial $P_i(t)$ defined as above for $\mathfrak{X}_{\mathfrak{p}}$ is equal to the i -th Betti number of the complex variety $X_{\mathbb{C}}$ obtained from X by base change via the given morphism $\phi^* : \text{Spec } \mathbb{C} \rightarrow \text{Spec } L$; we will abuse notation and refer to these as Betti numbers of X .

Now when C is a smooth projective curve over \mathbb{F}_q , geometric invariant theory (GIT) can be used to define the moduli space $\mathcal{M}^C(n, d)$ of stable, rank n , degree d vector bundles over C , at least for large enough q (we will review this construction in section §5). When n and d are coprime, the resulting moduli space $\mathcal{M}^C(n, d)$ is again a smooth projective variety over \mathbb{F}_q (or a suitable finite field extension).

It is not hard to see that the topology of the moduli space $\mathcal{M}(n, d)$ of stable vector bundles of rank n and degree d over a compact Riemann surface (or nonsingular complex projective curve) depends only on the genus of the Riemann surface, not on its complex structure. Thus, in order to study the cohomology of $\mathcal{M}(n, d)$, we can begin with a smooth projective curve $\pi : \mathfrak{C} \rightarrow B$ over an open subscheme B of the ring of integers in a number field $L \hookrightarrow \mathbb{C}$. Assuming that π admits a section, Seshadri’s extension of GIT to general base schemes (see [Ses77]) can be used to construct a moduli space of stable vector bundles of coprime rank and degree over \mathfrak{C} (see [Mar78] or [Gas97] Théorème 4.3). Given a prime p for which there exists a morphism $\text{Spec } \mathbb{F}_q \rightarrow B$ (for q a power of p) whose image we denote by \mathfrak{p} , base change produces a scheme isomorphic to the moduli scheme of stable bundles over a nonsingular projective curve $\mathfrak{C}_{\mathfrak{p}}$ over \mathbb{F}_q ; similarly, using the inclusion of the generic point $\text{Spec } L \hookrightarrow B$ and the given map $\text{Spec } \mathbb{C} \rightarrow \text{Spec } L$, base change produces a scheme isomorphic to the moduli scheme of stable bundles over a nonsingular projective curve over \mathbb{C} . Thus the Weil conjectures allow us as above to compute the Betti numbers of $\mathcal{M}(n, d)$ from the numbers of points defined over \mathbb{F}_{q^r} (for $r = 1, 2, \dots$) of $\mathcal{M}^C(n, d)$ where $C = \mathfrak{C}_{\mathfrak{p}}$ (cf. [HN75] pp. 239-242).

In fact it is technically easier to work with the moduli spaces $\mathcal{M}_{\Lambda}(n, d)$ and $\mathcal{M}_{\Lambda}^C(n, d)$ of stable vector bundles of rank n and fixed determinant line bundle Λ (of degree d); when n and d are coprime, calculating the Betti numbers of $\mathcal{M}_{\Lambda}(n, d)$ is equivalent to calculating those of $\mathcal{M}(n, d)$ since the determinant map $\mathcal{M}(n, d) \rightarrow \mathcal{M}(1, d)$ defined by $E \mapsto \det(E)$ is a fibration with fibre at $\Lambda \in \mathcal{M}(1, d)$ given by $\mathcal{M}_{\Lambda}(n, d)$, and it induces an isomorphism

$$H^*(\mathcal{M}(n, d)) \cong H^*(\mathcal{M}_{\Lambda}(n, d)) \otimes H^*(\mathcal{M}(1, d)).$$

Here $\mathcal{M}(1, d)$ is isomorphic to the Jacobian variety of C , which is an abelian

variety of dimension equal to the genus g of C . In this context, counting the number of \mathbb{F}_{q^r} -points of $\mathcal{M}_\Lambda^C(n, d)$ amounts to counting isomorphism classes of stable rank n vector bundles defined over \mathbb{F}_{q^r} on the curve C with fixed determinant line bundle Λ .

Tamagawa Numbers

Let C be a smooth projective curve over a finite field \mathbb{F}_q . Harder, Narasimhan, Desale and Ramanan (see [HN75, DR75]) showed how to count the number of \mathbb{F}_{q^r} -points of the moduli space $\mathcal{M}_\Lambda^C(n, d)$ of stable rank n degree d vector bundles over C with fixed determinant Λ of degree d on C , when n and d are coprime. In order to do this, they made use of the Tamagawa number of SL_n . Let us recall the relationship between the Tamagawa number of SL_n and moduli spaces of vector bundles.

Suppose $K = \mathbb{F}_q(C)$ is the function field of C . Recall that the adèle ring \mathbb{A}_K of K is defined as follows. If x is a closed point of C , denote the local ring at x by $\mathcal{O}_{X,x}$. Denote the completion of the local ring at x by $\hat{\mathcal{O}}_x$ and the field of fractions of $\hat{\mathcal{O}}_x$ by $\hat{\mathcal{K}}_x$. Then $\hat{\mathcal{O}}_x$ is a compact topological ring, and choosing a local parameter determines an isomorphism to $\mathbb{F}_{q^r}[[t]]$, where r is a strictly positive integer. Similarly, $\hat{\mathcal{K}}_x$ is a locally compact field isomorphic to $\mathbb{F}_{q^r}((t))$. For any finite set S of closed points of C , we define \mathbb{A}_S to be the product

$$\mathbb{A}_S = \prod_{x \in S} \hat{\mathcal{K}}_x \times \prod_{x \in C-S} \hat{\mathcal{O}}_x,$$

where on the right hand side, the notation $x \in C - S$ should be read “ x is a closed point of $C - S$.” Observe that each \mathbb{A}_S is a locally compact topological ring. The sets S are partially ordered by inclusion and we let \mathbb{A}_K be the locally compact topological ring obtained by taking the colimit of the rings \mathbb{A}_S as S varies.

Since SL_n is defined over $\text{Spec } \mathbb{Z}$, we can consider the set $SL_n(\mathbb{A}_K)$. This set can be equipped naturally with the structure of a locally compact topological group. Let \mathfrak{K} be the maximal compact subgroup of $SL_n(\mathbb{A}_K)$ which is the product $\prod_x SL_n(\hat{\mathcal{O}}_x)$ (again running over closed points of C). The group $SL_n(K)$ can be viewed as a discrete subgroup of $SL_n(\mathbb{A}_K)$. Let $\text{Bun}_{SL_n}(\mathbb{F}_q)$ denote the set of isomorphism classes of vector bundles on C , defined over \mathbb{F}_q , with trivial determinant. The starting point for the relationship between moduli spaces of bundles over C and Tamagawa numbers was Weil’s construction of a canonical bijection:

$$\mathfrak{K} \backslash SL_n(\mathbb{A}_K) / SL_n(K) \xrightarrow{\sim} \text{Bun}_{SL_n}(\mathbb{F}_q).$$

Briefly, any vector bundle \mathcal{E} on C is Zariski locally trivial. Thus, \mathcal{E} can be trivialized at the generic point η of C . Similarly, \mathcal{E} can be trivialized over the formal disc $\text{Spec } \hat{\mathcal{O}}_x$ for any $x \in C$. Elements of $SL_n(\mathbb{A}_K)$ can then be identified with collections $(\mathcal{E}, \varphi_\eta, \{\varphi_x\}_{x \in C})$ consisting of a vector bundle \mathcal{E} over C with trivial determinant, a trivialization φ_η of \mathcal{E} at the generic point, and a trivialization φ_x of \mathcal{E} over the formal disc at every closed point $x \in C$. Given an element

$g \in SL_n(\mathbb{A}_K)$, the points $x \in C$ at which g_x does not lie in $SL_n(\hat{\mathcal{O}}_x)$ form a closed subscheme S of C . We can reconstruct the vector bundle \mathcal{E} from g by twisting a trivial bundle on $C - S$ by the g_x for $x \in S$. For a fixed determinant line bundle $\Lambda \neq \mathcal{O}$ we simply replace \mathfrak{K} with a different maximal compact subgroup of $SL_n(\mathbb{A}_K)$ and replace $\text{Bun}_{SL_n}(\mathbb{F}_q)$ with the set $\text{Bun}_{SL_n}^\Lambda(\mathbb{F}_q)$ of isomorphism classes of vector bundles on C , defined over \mathbb{F}_q , with fixed determinant Λ . One may also consider the double coset space $\mathfrak{K} \backslash SL_n(\mathbb{A}_K) / SL_n(K)$ with SL_n replaced by other groups. The corresponding double coset space plays a key motivating rôle in the geometric Langlands program (see e.g. [Gai03, Fre]).

As the group $SL_n(\mathbb{A}_K)$ is locally compact it possesses a right invariant Haar measure, which is determined up to scalars. In fact, given a right invariant, non-zero differential form ω of top degree on SL_n (thought of as a group scheme over K), there exists a procedure to construct a uniquely determined right invariant measure $\omega_{\mathbb{A}_K}^\tau$ on $SL_n(\mathbb{A}_K)$. This measure induces a measure on the coset space $SL_n(\mathbb{A}_K) / SL_n(K)$ and we set

$$\tau(SL_n) = \int_{SL_n(\mathbb{A}_K) / SL_n(K)} \omega_{\mathbb{A}_K}^\tau.$$

The crucial fact for our discussion is the fact due essentially to Siegel, but explicitly proved by Weil, that the Tamagawa number $\tau(SL_n) = 1$ (see [Wei82] Theorem 3.3.1).

The differential form ω induces measures on the compact groups $SL_n(\hat{\mathcal{O}}_x)$ for each $x \in C$, and thus on \mathfrak{K} . The connection between the Tamagawa number and isomorphism classes of bundles is provided by Siegel's mass formula:

$$\tau(SL_n) = \text{vol}(\mathfrak{K}) \sum_{\mathcal{E} \in \text{Bun}_{SL_n}^\Lambda(\mathbb{F}_q)} \frac{1}{|\text{Aut}(\mathcal{E})|}.$$

If we let $\zeta_C(s) = Z_C(q^{-s})$, the volume of \mathfrak{K} can be computed explicitly in terms of the zeta function of C :

$$\text{vol}(\mathfrak{K}) = q^{-(n^2-1)(g-1)} \zeta_C(2)^{-1} \cdots \zeta_C(n)^{-1}.$$

(For generalizations of this result to other groups G — and to the case of smooth reductive group schemes over C — we refer the reader to the work of Behrend and Dhillion [BDa, BDb].)

The automorphisms of a stable bundle over \mathbb{F}_q are simply given by multiplication by nonzero scalars, i.e. elements of the multiplicative group $\mathbb{G}_m(\mathbb{F}_q)$ of \mathbb{F}_q . The set $\mathfrak{K} \backslash SL_n(\mathbb{A}_K) / SL_n(K) \cong \text{Bun}_{SL_n}^\Lambda(\mathbb{F}_q)$ can be partitioned by Harder-Narasimhan type into subsets $\text{Bun}_{SL_n}^{\Lambda, \mu}(\mathbb{F}_q)$. Via the map $SL_n(\mathbb{A}_K) / SL_n(K) \rightarrow \mathfrak{K} \backslash SL_n(\mathbb{A}_K) / SL_n(K)$, this induces a stratification of the coset space $SL_n(\mathbb{A}_K) / SL_n(K)$. It follows from the Siegel mass formula that when n and d are coprime the number of isomorphism classes of stable (equivalently semistable) vector bundles on C with rank n and fixed determinant Λ of degree d is given by

$$q^{(n^2-1)(g-1)} \zeta_C(2) \cdots \zeta_C(n) - \sum_{\mu \neq (d/n, \dots, d/n)} \sum_{\mathcal{E} \in \text{Bun}_{SL_n}^{\Lambda, \mu}(\mathbb{F}_q)} \frac{1}{|\text{Aut}(\mathcal{E})|},$$

where as before μ is the Harder-Narasimhan type of \mathcal{E} determined by the ranks n_j and degrees d_j of the subquotients of the Harder-Narasimhan filtration of \mathcal{E} .

Summarizing, we get an inductive formula for the sum

$$\sum_{\mathcal{E} \text{ semistable of rank } n \text{ and degree } d} \frac{1}{|Aut(\mathcal{E})|}$$

where \mathcal{E} runs over the set of isomorphism classes of semistable vector bundles on C , defined over \mathbb{F}_q , of rank n and fixed determinant Λ of degree d . When n and d are coprime it enables us to calculate the Betti numbers of $\mathcal{M}_\Lambda(n, d)$ and $\mathcal{M}(n, d)$ via the Weil conjectures. This leads to an inductive procedure for calculating the Betti numbers which is formally the same as that obtained by Atiyah and Bott via equivariant Morse theory.

3 Equivariant motivic cohomology

In the next two sections our aim is to extend the circle of ideas discussed above to the framework of motives, and motivic homotopy theory as introduced by Morel and Voevodsky (see [Voe00, MVW06, MV99]). In order to do this, we need a version of equivariant motivic cohomology for a linear algebraic group acting on a projective variety. In the subsequent two sections we will apply this theory to the study of the motivic cohomology of GIT quotients and of moduli spaces of vector bundles over a nonsingular projective curve. In this section, we discuss the construction and basic properties of equivariant motivic cohomology; for more details see [ADK] (or its predecessor [EG98] for a definition of equivariant motivic cohomology for smooth varieties using Bloch's higher Chow groups, and the homological version for arbitrary varieties).

Throughout this section and the next, k will denote a perfect field of arbitrary characteristic, and all varieties, groups and schemes will be assumed to be defined over k . If G is a linear algebraic group over k , a variety (respectively, a projective variety) X equipped with an algebraic G -action will be called G -quasiprojective if it admits an ample G -equivariant line bundle.

3.1 The Borel construction

To define an equivariant motivic cohomology theory, we emulate the Borel construction in topology. (We only discuss the theory for smooth schemes here; it is possible to develop a good theory for schemes with mild singularities, e.g., seminormal schemes.) The construction of an algebraic model for BG described here is essentially that of [Tot99, MV99]. Let G be a linear algebraic group over k . If $\rho : G \rightarrow GL(V)$ is any faithful k -rational representation of V , we can define a space $EG(\rho)$ as follows. Consider the affine space $V^{\oplus n}$ with its natural G -action. If n is sufficiently large, then $V^{\oplus n}$ contains an open subscheme V_n on which G acts freely and such that the quotient V_n/G exists as a smooth quasiprojective k -variety. The natural map $V^{\oplus n} \hookrightarrow V^{\oplus n+1}$ induces a G -equivariant map $V_n \hookrightarrow V_{n+1}$ and hence a morphism $V_n/G \rightarrow V_{n+1}/G$. We then define $BG(\rho)$

to be the ind-scheme $\operatorname{colim}_n V_n/G$ and $EG(\rho) = \operatorname{colim}_n V_n$. Similarly, if we let X be a smooth G -quasiprojective variety, then we get an inductive system of spaces $V_n \times_G X$ and we set $X_G(\rho) = \operatorname{colim}_n V_n \times_G X$.

Let $\mathcal{S}hv_{\text{Nis}}(\mathcal{S}m/k)$ be the category of Nisnevich sheaves of sets on $\mathcal{S}m/k$; for brevity we will denote this category by $\mathcal{S}pc$ and refer to its objects as spaces. The assignment $X \mapsto \operatorname{Hom}_{\mathcal{S}m/k}(\cdot, X)$, sending a scheme to its functor of points, determines a fully faithful embedding $\mathcal{S}m/k \rightarrow \mathcal{S}pc$. This follows from the Yoneda lemma together with the observation that the Nisnevich topology is subcanonical (i.e., every representable presheaf is a sheaf). Thus, given a smooth scheme X , when we refer to “the space X ,” we will mean the corresponding functor. In a similar way, every ind-scheme can be viewed as a Nisnevich sheaf of sets and we consider the ind-schemes $X_G(\rho)$ as Nisnevich sheaves.

The motivic homotopy category $\mathcal{H}(k)$ (respectively the pointed motivic homotopy category $\mathcal{H}_*(k)$) can be constructed from $\mathcal{S}pc$ (respectively the category $\mathcal{S}pc_*$ of pointed spaces) by localization at an appropriate class of weak equivalences in the sense of model category theory. In analogy with classical homotopy theory, objects of the (pointed) motivic homotopy category are (pointed) spaces and morphisms are (pointed) “ \mathbb{A}^1 -homotopy classes of maps,” appropriately defined. We will minimize explicit definitions involving terms whose definitions are formally analogous to those from classical homotopy theory. For example, a space X will be called \mathbb{A}^1 -contractible if it is equivalent to $\operatorname{Spec} k$ in $\mathcal{H}(k)$. We remark that a Zariski locally trivial smooth morphism $f : X \rightarrow Y$ of smooth schemes with \mathbb{A}^1 -contractible fibres (e.g., affine space fibres) is an \mathbb{A}^1 -weak equivalence (see [MV99] for a precise definition of \mathbb{A}^1 -weak equivalence and [AD07] for more detailed discussion of this example). The space $X_G(\rho)$ gives rise to an object in $\mathcal{H}(k)$ or $\mathcal{H}_*(k)$. As one expects, the space $X_G(\rho)$ viewed as an object of $\mathcal{H}_*(k)$ is independent of the choice of faithful representation ρ .

Proposition 3.1 *For any ρ , the space $EG(\rho)$ is \mathbb{A}^1 -contractible in $\mathcal{H}(k)$. For any two faithful representations ρ, ρ' of G , there is a canonical isomorphism $X_G(\rho) \cong X_G(\rho')$ in $\mathcal{H}(k)$.*

Henceforth, we write X_G for the object in the motivic homotopy category defined by $X_G(\rho)$ for *any* faithful k -rational representation ρ . The space X_G will be called the *motivic Borel construction* for G acting on X . The proof of the proposition above is conceptually very simple and involves a “space level” version of the Bogomolov double fibration construction (see [EG98] Proposition-Definition 1). Indeed, for two faithful representations ρ, ρ' , we construct a double inductive system of spaces $X_{n,n'}$ which map to both $V_n \times_G X$ and $V_{n'} \times_G X$, where $V_n \times_G X$ denotes the geometric quotient variety of $V_n \times X$ by the free action of G . Using [MV99] §4 Prop 2.3, and basic results about commutation of colimits, we show that as both $n, n' \rightarrow \infty$, the space $X_{n,n'}$ becomes weakly equivalent to $X_G(\rho)$ and $X_G(\rho')$.

Thus, mimicking the definition of ordinary equivariant cohomology, and given the definition of motivic cohomology, one can make the following definition of equivariant motivic cohomology.

Definition 3.2 *The equivariant motivic cohomology $H_G^{\bullet,\bullet}(X, \mathbb{Z})$ of a smooth G -quasiprojective scheme X is defined by the equality*

$$H_G^{\bullet,\bullet}(X, \mathbb{Z}) = H^{\bullet,\bullet}(X_G, \mathbb{Z}) = H^{\bullet,\bullet}(X_G(\rho), \mathbb{Z}) \quad (9)$$

for any faithful representation ρ of G .

Proposition 3.1 together with a modification of [Voe03] Prop 6.1 then gives rise to the following result which asserts that motivic cohomology may be computed as a limit of motivic cohomologies of approximations to the Borel construction.

Proposition 3.3 *For any faithful representation ρ of G , we have an isomorphism*

$$H^{\bullet,\bullet}(X_G(\rho), \mathbb{Z}) \cong \lim_n H^{\bullet,\bullet}(V_n \times_G X, \mathbb{Z}). \quad (10)$$

One key ingredient in the proof of this result, which will be extremely useful when studying moduli spaces of bundles, is the following standard consequence of the existence of Gysin sequence regarding excising subvarieties from a smooth variety. Essentially by definition, the motivic cohomology groups $H^{p,q}(X, \mathbb{Z})$ vanish for $q < 0$. Basic results about Gysin triangles then give rise to the following result.

Lemma 3.4 *Suppose X is a smooth scheme and Z is a closed subscheme of codimension d in X . Then the restriction map*

$$H^{p,q}(X, \mathbb{Z}) \longrightarrow H^{p,q}(X - Z, \mathbb{Z})$$

is an isomorphism whenever $q < d$. In this situation, we will say that the restriction map is an isomorphism on motivic cohomology of weight $q < d$.

Remark 3.5 *The motivic cohomology of a smooth scheme X can be described as the set of \mathbb{A}^1 -homotopy classes from X to $K(\mathbb{Z}(q), p)$, where $K(\mathbb{Z}(q), p)$ are motivic Eilenberg-MacLane spaces. From this point of view, contravariant functoriality for equivariant motivic cohomology of arbitrary G -equivariant morphisms of smooth schemes is evident. Furthermore, for different “motivic spectra,” this approach allows a definition of “Borel-style” generalized equivariant algebraic cohomology theories, for example Borel-style equivariant algebraic K -theory or equivariant algebraic cobordism. In particular, if the motivic spectrum is a motivic ring spectrum, the associated cohomology theory has a ring structure; this is known for motivic cohomology.*

3.2 Basic properties

It follows immediately from Voevodsky’s comparison theorem (see [Voe02] or [MVW06]) and Proposition 3.3 that the equivariant motivic cohomology we have defined is isomorphic to the equivariant higher Chow groups defined by

Edidin-Graham in [EG98]. (Note that equivariant higher Chow groups constitute a Borel-Moore homology theory, and hence the isomorphism just mentioned can be viewed as a form of duality between compactly supported homology and cohomology.) All natural properties of motivic cohomology (see [Voe00] §3) immediately adapt to equivariant motivic cohomology: one has homotopy invariance for morphisms of schemes which are \mathbb{A}^1 -weak equivalences, equivariant Mayer-Vietoris sequences, equivariant Thom-Gysin sequences and projective bundle formulae. As other examples of properties of Borel-style equivariant cohomology that adapt to the motivic setting, we have the following results.

Suppose that H is a closed subgroup of G . If X is an H -quasi-projective scheme, we write $G \times_H X$ for the twisted product space, i.e., the quotient of $G \times X$ by the action of H defined by $h \cdot (g, x) = (gh^{-1}, h \cdot x)$. (There seem to be many different ways to denote this space in the literature. We follow the standard topological convention and hope the reader does not confuse this with fiber product constructions.)

Lemma 3.6 *Suppose that X is a smooth H -quasiprojective variety and suppose that G is a connected linear algebraic group such that $H \subset G$. Let Y denote the twisted product space $G \times_H X$. Then there is a canonical isomorphism $X_H \cong Y_G$ in $\mathcal{H}(k)$.*

Lemma 3.7 *Let X be a smooth G -quasiprojective variety with a scheme-theoretically free G -action. Then the quotient X/G exists as a smooth scheme and the natural map $X_G \rightarrow X/G$ is an isomorphism in $\mathcal{H}(k)$.*

The first lemma is proved in essentially the same way as Proposition 3.1. The second lemma follows from [MV99] §4 Lemma 2.8.

We write $\mathbf{DM}_{Nis}^{eff,-}(k, \mathbb{Q})$ for the derived category of effective rational motivic complexes over k . There is a covariant functor $\mathcal{S}m/k \rightarrow \mathbf{DM}_{Nis}^{eff,-}(k, \mathbb{Q})$ defined by $X \mapsto M(X)$ (see [MVW06] Definition 14.1) where $M(X)$ is called the rational motive of X . The object $M(X)$ can be thought of as analogous to the singular chain complex (with rational coefficients) of the infinite symmetric product of a topological space M , viewed as an object in the derived category of \mathbb{Q} -vector spaces. Rational motivic cohomology groups can also be computed as $Hom_{\mathbf{DM}_{Nis}^{eff,-}(k, \mathbb{Q})}(M(X), \mathbb{Q}(q)[p])$ where $\mathbb{Q}(q)$ is a certain complex of sheaves in $\mathbf{DM}_{Nis}^{eff,-}(k, \mathbb{Q})$. In everything that follows, we work with motives and motivic cohomology with *rational coefficients*.

In this setting, Lemma 3.7 can be generalized to varieties with finite quotient singularities.

Proposition 3.8 *Let X be a smooth G -quasi-projective variety such that (i) G acts on X with finite stabilizers and (ii) a geometric quotient X/G exists as a scheme. The projection morphism $X_G \rightarrow X/G$ then gives an isomorphism in motivic cohomology*

$$H^{\bullet,\bullet}(X/G, \mathbb{Q}) \xrightarrow{\sim} H_G^{\bullet,\bullet}(X, \mathbb{Q}).$$

We also have a version of the Leray-Hirsch theorem. Recall that the topological Leray-Hirsch theorem allows one to compute the (additive) structure of the cohomology of certain fibre bundles (where the cohomology of the fibre is a free module over the cohomology of a point). Analogously, we have the following result.

Proposition 3.9 *Let G be a split connected reductive group over k . Let P be a parabolic subgroup defined over k . Suppose that $\mathcal{P} \rightarrow X$ is a principal G -bundle on a smooth scheme X which has a k -rational point. Then*

- $H^{\bullet,\bullet}(G/P, \mathbb{Q})$ is free as a module over the motivic cohomology of $\mathrm{Spec} k$, and
- given elements c_α of $H^{\bullet,\bullet}(\mathcal{P} \times_G G/P, \mathbb{Q})$ whose restriction to G/P (the fibre over a fixed k -rational point) form a basis for $H^{\bullet,\bullet}(G/P, \mathbb{Q})$ (such elements always exist), one can construct an isomorphism of modules over the motivic cohomology of $\mathrm{Spec} k$:

$$H^{\bullet,\bullet}(\mathcal{P} \times_G G/P, \mathbb{Q}) \cong H^{\bullet,\bullet}(X, \mathbb{Q}) \otimes_{H^{\bullet,\bullet}(\mathrm{Spec} k, \mathbb{Q})} H^{\bullet,\bullet}(G/P, \mathbb{Q}).$$

Let $T \subset B \subset G$ be a choice of a maximal torus T inside a Borel subgroup B of a split connected reductive group G and suppose that X is a smooth G -quasi-projective scheme. Fix a faithful k -rational representation ρ of G , and consider the corresponding models $X_G(\rho)$. Observe that ρ gives faithful k -rational representations of B and T as well (by restriction) and we obtain maps $X_T(\rho) \rightarrow X_B(\rho) \rightarrow X_G(\rho)$, functorial in X . The morphism $X_T(\rho) \rightarrow X_B(\rho)$ can be checked to be Zariski locally trivial with fibres isomorphic to affine space and is thus an \mathbb{A}^1 -weak equivalence. If we assume furthermore that X has a k -rational point, then the fibre of $X_B(\rho) \rightarrow X_G(\rho)$ over any such point is isomorphic to G/B . The identification $X_T \cong (G/T \times X)_G$ induces a natural action of the Weyl group W on X_T . Tracking the action of the Weyl group and applying the previous proposition, we obtain the following result.

Theorem 3.10 *Suppose that X is a smooth G -quasi-projective scheme possessing a k -rational point. The natural map $X_T \rightarrow X_G$ induces an isomorphism of rings*

$$H_G^{\bullet,\bullet}(X, \mathbb{Q}) \rightarrow H_T^{\bullet,\bullet}(X, \mathbb{Q})^W. \quad (11)$$

Example 3.11 *If T is a split torus, we can fix an isomorphism $T \cong (\mathbb{G}_m)^n$ where $n = \mathrm{rk} T$ and \mathbb{G}_m is the multiplicative group of the field k . One can check that $B\mathbb{G}_m$ is isomorphic to an infinite dimensional projective space and that BT is isomorphic to a product of $\mathrm{rk} T$ copies of an infinite dimensional projective space. Thus, using the projective bundle theorem and the Künneth formula, one can show that $M(BT) \cong \bigotimes_{i=1}^n (\bigoplus_{p \geq 0} \mathbb{Z}(p)[2p])$.*

4 Perfection and equivariant perfection of stratifications

4.1 A schematic review of the cohomology of GIT quotients

Mumford’s geometric invariant theory [MFK94] gives a method to construct and study quotients of certain reductive group actions on algebraic varieties. To fix ideas, consider a complex reductive group G , a smooth complex projective variety X equipped with an algebraic action of G and a G -equivariant very ample line bundle L on X . In this situation, Mumford introduced a natural G -invariant open subset of ‘semistable’ points $X^{ss} \subset X$ for which a projective (categorical) quotient variety exists; this quotient will be denoted $X//G$. In general $X//G$ is not an orbit space for the action of G on X^{ss} ; however X^{ss} contains an open subset X^s of ‘stable’ points for the linear action such that the image of the restriction to X^s of the quotient map $X^{ss} \rightarrow X//G$ is an open subset of $X//G$ which can be identified naturally with X^s/G . Mumford showed that stability and semistability of a point $x \in X$ can be tested via one-parameter subgroups $\lambda : \mathbb{G}_m \rightarrow G$ where \mathbb{G}_m is the multiplicative group of \mathbb{C} (the Hilbert-Mumford criterion); thus there is an effective way to identify the sets of stable and semistable points.

The complement of X^{ss} is called the set of unstable points and denoted X^{us} . To the choice of linearization, there is a naturally associated “instability” stratification of X^{us} (see [Kem78, Hes78, Kir84]). By taking X^{ss} to be an open stratum, this extends to a stratification of X .

Remark 4.1 *Suppose that $T \subset G$ is a maximal torus. The cocharacter group $X_*(T)$ is naturally a \mathbb{Z} -module and thus we can form the tensor product $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Weyl group W acts on $X_*(T)$ by conjugation. Technically, to define the stratification, one specifies a W -invariant norm q on $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, but this choice will be unimportant for our purposes.*

It was shown in [Kir84] that this stratification $\{S_\beta : \beta \in \mathcal{B}\}$, where the indexing set \mathcal{B} is a finite set of co-adjoint orbits of G , has the following properties:

- P)** The stratification is rationally G -equivariantly perfect, so that there is a (\mathbb{Q} -vector space) isomorphism of equivariant cohomology groups

$$H_G^j(X, \mathbb{Q}) \cong \bigoplus_{\beta \in \mathcal{B}} H_G^{j-2d_\beta}(S_\beta, \mathbb{Q}) \quad (12)$$

for all $j \geq 0$, where d_β is the (complex) codimension of S_β in X .

- S1)** The stratum indexed by $0 \in \mathcal{B}$ coincides with the locus X^{ss} of semistable points of X for the linear G -action.

S2) If $\beta \in \mathcal{B} \setminus \{0\}$ then there is a nonsingular subvariety Z_β of X , a reductive subgroup L_β of G and a linear action of L_β on Z_β , with corresponding semistable locus denoted Z_β^{ss} , such that

$$H_G^\bullet(S_\beta, \mathbb{Q}) \cong H_{L_\beta}^\bullet(Z_\beta^{ss}, \mathbb{Q}). \quad (13)$$

Remark 4.2 See Theorem 4.7 for more discussion of the scheme structure of this stratification.

This provides us with an inductive procedure for calculating the G -equivariant Betti numbers $\dim H_G^j(X^{ss}, \mathbb{Q})$ of X^{ss} . When $X^{ss} = X^s$ (so that the GIT quotient $X//G$ coincides with the orbit space X^{ss}/G and G acts with only finite stabilizers on X^{ss}) one observes that

$$H_G^\bullet(X^{ss}, \mathbb{Q}) \cong H^\bullet(X//G, \mathbb{Q}). \quad (14)$$

Moreover the Leray spectral sequence for rational cohomology associated with the fibration

$$X \times_G EG \longrightarrow BG$$

degenerates because X is smooth and projective (see [Del74]), and one obtains an isomorphism of rational vector spaces

$$H_G^\bullet(X, \mathbb{Q}) \cong H^\bullet(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^\bullet(BG, \mathbb{Q}). \quad (15)$$

Thus we obtain a method for calculating the Betti numbers of $X//G$ in terms of the Betti numbers of X and certain smooth projective subvarieties which turn up inductively, together with the classifying spaces of G and certain reductive subgroups of G ; indeed using versions of Theorem 3.10 and the Bialynicki-Birula decomposition (see Theorem 4.4) for ordinary cohomology, we can reduce this to studying the Weyl group action on the cohomology of the classifying spaces of a maximal torus T of G and subtori of T , together with components of their fixed point sets on X .

An alternative method for obtaining an equivalent inductive procedure enabling us to calculate the Betti numbers of $X//G$, at least when G acts freely on X^{ss} , is provided by the Weil conjectures: the stratification allows us to count the semistable points of associated varieties defined over finite field as the total number of points minus the sum over $\beta \in \mathcal{B}$ of the number of points in the stratum labelled by β (see [Kir84]§15 for more details).

In the remainder of this section, we will show how to adapt the stratification just described to study the motivic cohomology of GIT quotients.

Remark 4.3 These techniques for computing the cohomology of the quotient variety $X//G$ when $X^{ss} = X^s$ were refined and extended to cover $X^{ss} \neq X^s$ in a series of papers [Kir85, Kir86c, Kir87, JK95, JK98, JKKW03].

4.2 The Bialynicki-Birula stratification

Let k be a perfect field, and let \mathbb{G}_m be the multiplicative group over k . Let X be a smooth \mathbb{G}_m -projective algebraic k -variety. The fixed-point locus $X^{\mathbb{G}_m}$ of the \mathbb{G}_m -action of X is in general disconnected, though smooth, and we denote by $\{Z_i : i \in I\}$ the set of its connected components. There is a stratification of X indexed by I whose properties are summarized in the following theorem.

Theorem 4.4 ([BB73],[Hes81]) *Let $\{Z_i : i \in I\}$ be the set of connected components of $X^{\mathbb{G}_m}$. There is a stratification of X by \mathbb{G}_m -stable, smooth, locally closed subvarieties $\{Y_i : i \in I\}$ together with morphisms $Y_i \rightarrow Z_i$ for $i \in I$ which are \mathbb{G}_m -equivariant vector bundles. The inclusion $Z_i \hookrightarrow X$ factors through the zero section of the bundle $Z_i \hookrightarrow Y_i$ and the inclusion $Y_i \hookrightarrow X$ for each $i \in I$.*

As observed by Brosnan (see [Bro05] and the references therein), the Bialynicki-Birula decomposition actually gives a decomposition of the integral motive of X . Thus, one obtains the following result.

Theorem 4.5 *Suppose that X is a smooth \mathbb{G}_m -projective algebraic variety over k . With notation as in Theorem 4.4, if c_i denotes the codimension of Z_i , then one has a decomposition*

$$H^{\bullet, \bullet}(X, \mathbb{Z}) \cong \bigoplus_{i \in I} H^{\bullet - 2c_i, \bullet - c_i}(Z_i, \mathbb{Z})$$

of modules over the motivic cohomology of $\text{Spec } k$.

We can also use the Bialynicki-Birula decomposition to prove a fixed-point localization theorem in the setting of \mathbb{G}_m -equivariant motivic cohomology, in the same style as [Kir84] (see [ADK] for more details). Furthermore, the decomposition given above holds in any oriented algebraic cohomology theory (cf. [NZ06]).

4.3 The instability stratification

Let G be a split reductive group over k and let X be a smooth G -projective variety. We fix a very ample G -linearized line bundle L on X and we write $X^{us} = X^{us}(L)$ for the complement of the semistable locus $X^{ss} = X^{ss}(L)$. In addition, we write $X \hookrightarrow \mathbb{P}(V)$ for the projective embedding determined by L . Henceforth, we will suppress L .

Next we need more detail about the stratification of X^{us} than was discussed in §4.1.

If $T \subset G$ is a split maximal torus of G , then we let $\mathbf{X}^*(T)$ denote the character group of T and $\mathbf{X}_*(T)$ the cocharacter group of T .

Remark 4.6 *The assumption of splitness here is made for simplicity; one can prove versions of the theorems below without this assumption in place.*

Theorem 4.7 *There is a natural decomposition of X into G -invariant subvarieties S_β labelled by a finite subset \mathcal{B} of $\mathbf{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ which has the following properties. To each $\beta \in \mathcal{B}$ there is a canonically associated parabolic subgroup $P_\beta \subset G$ with a k -defined Levi subgroup $L_\beta \subset P_\beta$ such that*

- *there is a smooth L_β -stable closed subvariety $Z_\beta \subset X$, which is a component of the fixed point locus of a one-parameter subgroup of T representing β , and*
- *there is a P_β -stable subvariety $Y_\beta \subset X$ and an L_β -equivariant surjective morphism $Y_\beta \rightarrow Z_\beta$, which is Zariski locally trivial with fibres isomorphic to affine spaces.*

Moreover there is a linearization of the induced L_β -action on Z_β such that if Z_β^{ss} denotes the semistable locus for this L_β -action on Z_β and Y_β^{ss} denotes the fibre product of Y_β and Z_β^{ss} over Z_β then S_β is the scheme-theoretic image of $G \times_{P_\beta} Y_\beta^{ss}$ under the multiplication morphism $m_\beta : G \times_{P_\beta} Y_\beta^{ss} \rightarrow X$.

X is the disjoint union of the subvarieties S_β and X^{ss} coincides with the subvariety S_0 labelled by $0 \in \mathcal{B}$. Furthermore, one can totally order \mathcal{B} so that $\{S_\beta : \beta \in \mathcal{B}\}$ is a stratification in the sense that $S_\beta \subseteq \cup_{\beta' \geq \beta} S_{\beta'}$ for each $\beta \in \mathcal{B}$. Finally

- *the morphism $m_\beta : G \times_{P_\beta} Y_\beta^{ss} \rightarrow S_\beta$ is a finite, G -equivariant, birational, surjective morphism, and hence an equivariant resolution of singularities, and*
- *if, in addition, the action of G on X has the property that for any $\beta \in \mathcal{B}$ and any point $y \in Z_\beta^{ss}$*

$$\mathrm{Lie}(P_\beta) = \{\xi \in \mathrm{Lie}(G) \mid \xi_y \in T_y Y_\beta\},$$

where $\xi_y \in T_y X$ denotes the tangent vector at y given by the infinitesimal action of ξ (we will say that the linearized action is manageable), then m_β is an isomorphism. This condition is always satisfied when the characteristic of k is 0.

Over an arbitrary field of characteristic 0, the result above was proved by Hesselink (see [Hes78, Hes81]). In particular, he showed that all actions in characteristic zero are manageable. A proof of the theorem for varieties over an arbitrary perfect field may be found in [ADK]. Without the manageability hypothesis counterexamples where m_β is not an isomorphism can be given even for (Frobenius twisted) SL_2 -actions on \mathbb{P}^1 over algebraically closed fields of characteristic $p > 0$.

4.4 The motivic cohomology of GIT quotients

With the notations of the previous section in place, we can state the main result which is a generalization of the main theorem of [Kir84]. Suppose that X

is a smooth G -projective variety over k . Let $\{S_\beta : \beta \in \mathcal{B}\}$ denote the instability stratification of X . According to Theorem 4.7, the indexing set of the instability stratification can be totally ordered so that $\bar{S}_\beta \subseteq \cup_{\beta' \geq \beta} S_{\beta'}$ for each $\beta \in \mathcal{B}$; we can choose an indexing function $f : \mathcal{B} \rightarrow \{0, 1, \dots, |\mathcal{B}| - 1\}$ commensurate with this total order. For simplicity, let us write S_i in place of S_β where $i = f(\beta)$. Let U^i denote the complement of \bar{S}_i in X for $0 \leq i \leq |\mathcal{B}| - 1$ and let $U^{|\mathcal{B}|} = X$. There is thus a diagram

$$U^i \hookrightarrow U^{i+1} \hookleftarrow S_i$$

where the left inclusion is a G -equivariant open immersion and the right inclusion is a G -equivariant closed immersion and S_i is the complement of U^i in U^{i+1} . If the linearized action is manageable, then S_i is in fact smooth and isomorphic to $G \times_{P_i} Y_i^{ss}$. This means that $(S_i)_G \cong (Y_i^{ss})_{P_i}$ and hence that

$$H_G^{\bullet, \bullet}(S_i, \mathbb{Z}) \cong H_{P_i}^{\bullet, \bullet}(Y_i^{ss}, \mathbb{Z}) \cong H_{L_i}^{\bullet, \bullet}(Z_i^{ss}, \mathbb{Z})$$

since P_i is \mathbb{A}^1 -homotopy equivalent to L_i and Y_i^{ss} is \mathbb{A}^1 -homotopy equivalent to Z_i^{ss} . We can therefore consider the equivariant Thom-Gysin sequence

$$\dots \longrightarrow H_{L_i}^{\bullet-2d_i, \bullet-d_i}(Z_i^{ss}, \mathbb{Q}) \longrightarrow H_G^{\bullet, \bullet}(U^i, \mathbb{Q}) \longrightarrow H_G^{\bullet, \bullet}(U^{i-1}, \mathbb{Q}) \longrightarrow \dots$$

The following result is a statement for motivic cohomology analogous to equivariant perfection.

Theorem 4.8 *Suppose that G is a split reductive group over k . Let X be a smooth G -projective algebraic k -variety with fixed ample G -equivariant line bundle L . Suppose that the linearized G -action on X is manageable. Let $\{S_\beta : \beta \in \mathcal{B}\}$ be the stratification of Theorem 4.7 with the indexing set \mathcal{B} identified with $\{0, 1, \dots, |\mathcal{B}| - 1\}$ as above. Then the Thom-Gysin long exact sequences of the inclusion $S_i \hookrightarrow U^i$ break up into short exact sequences of the form*

$$0 \longrightarrow H_G^{\bullet-2d_i, \bullet-d_i}(S_i, \mathbb{Q}) \cong H_{L_i}^{\bullet-2d_i, \bullet-d_i}(Z_i^{ss}, \mathbb{Q}) \longrightarrow H_G^{\bullet, \bullet}(U^i, \mathbb{Q}) \longrightarrow H_G^{\bullet, \bullet}(U^{i-1}, \mathbb{Q}) \longrightarrow 0.$$

Thus the equivariant cohomology of $X = U^{|\mathcal{B}|}$ can be “reconstructed” from the equivariant cohomology of $X^{ss} = U^0$ and the L_β -equivariant cohomologies of Z_β^{ss} , which are inductively of the same form.

The proof of this theorem can be obtained by applying the properties of motivic cohomology discussed in the previous section to modify the proof given in [Kir84]: this is due to the power and “topological” nature of motivic cohomology. Indeed, the result follows immediately from a motivic version of the “Atiyah-Bott lemma” (see [Kir84] Lemma 2.18 or [AB83] Prop 13.4) which guarantees that a certain equivariant motivic Euler class (see [Voe03] §4 for the definition) is not a zero divisor in rational motivic cohomology. The proof of this lemma involves, as in the Atiyah-Bott case, a reduction to maximal tori (via Theorem 3.10) and an identification of the composite of the Gysin map with restriction to Z_i^{ss} in the exact sequence of Theorem 4.8 with cupping with the Euler class of the normal bundle.

Theorem 4.8 enables us to compute inductively the equivariant motivic cohomology of X^{ss} , and thus using Proposition 3.8 to compute the motivic cohomology of the quotient $X//G$ when $X^{ss} = X^s$.

Corollary 4.9 *Let a reductive group G act on a smooth G -projective variety X with a fixed G -linearized line bundle L . Suppose that the conditions of Theorem 4.8 hold and that in addition $X^{ss} = X^s$. Then the inclusion $X^{ss} \hookrightarrow X$ induces a surjection*

$$H_G^{\bullet,\bullet}(X, \mathbb{Q}) \rightarrow H_G^{\bullet,\bullet}(X^{ss}, \mathbb{Q}) \cong H^{\bullet,\bullet}(X//G, \mathbb{Q}). \quad (16)$$

Remark 4.10 *When a reductive group G acts linearly on a smooth projective variety X with $X^{ss} \neq X^s$, we can still use Theorem 4.8 to compute inductively the equivariant motivic cohomology of X^{ss} . Then if d is the codimension of the complement of X^s in X^{ss} we have*

$$H^{i,j}(X^s/G, \mathbb{Q}) \cong H_G^{i,j}(X^s, \mathbb{Q}) \cong H_G^{i,j}(X^{ss}, \mathbb{Q})$$

when $j < d$ by Lemma 3.4.

Remark 4.11 *One can axiomatize the conditions required to make a version of Theorem 4.8 hold for generalized equivariant motivic cohomology theories in the sense of Remark 3.5. Closely related results have been obtained by Chai and Neeman (see [CN98]). The conditions are satisfied by, for example, by étale cohomology and Betti cohomology (see [ADK]).*

Remark 4.12 *The kernel of the surjection (16) can be studied in different ways. One way, modelled on the results of [Kir84], is to use Theorem 4.8. Another is to relate intersection theory on $X//G$ to equivariant intersection theory on $H_G^{\bullet,\bullet}(X, \mathbb{Q})$; we shall not pursue this here, but see, for example, [EG98, ES89, JK98].*

5 The motivic cohomology of moduli spaces of bundles over a curve

In this section, we show how the discussion of §3 and §4 can be brought to bear on the study of motivic cohomology for the moduli spaces $\mathcal{M}(n, d)$ of stable bundles of coprime rank n and degree d over a smooth projective curve C . We begin by recalling the GIT construction of $\mathcal{M}(n, d)$ following Newstead [New78]; the emphasis of our discussion is slightly different from existing treatments as we aim to keep the analogy between the algebraic and topological categories at the forefront.

To do this, let C be a smooth projective curve over field k which for simplicity we assume is algebraically closed. (The condition on k can be weakened, but this complicates discussion of some of the constructions.) In §3, we recalled an algebraic construction of the classifying space BGL_n . This space, uniquely

defined as an object in the \mathbb{A}^1 -homotopy category, was constructed as a limit of smooth quasi-projective varieties depending on the choice of a faithful representation ρ of GL_n . If we take ρ to be the standard n -dimensional representation of GL_n , then it is easy to see that we have (as ind-varieties)

$$BGL_n(\rho) = \operatorname{colim}_{\ell} Gr(\ell, n)$$

where $Gr(\ell, n)$ is the Grassmannian of linear n -dimensional quotients of a fixed ℓ -dimensional vector space. Henceforth, we suppress ρ and our main object of study will be the space

$$Map_d(C, BGL_n) \stackrel{\text{def}}{=} \operatorname{colim}_{\ell} Map_d(C, Gr(\ell, n))$$

of morphisms of degree d from C to BGL_n . Here the spaces $Map_d(C, Gr(\ell, n))$ are (not necessarily smooth) varieties which can be identified with subschemes of the Hilbert scheme of $C \times Gr(\ell, n)$.

We can construct the moduli space of semistable bundles on C , at least when d is sufficiently large, as a GIT quotient of (an open subscheme of) $Map_d(C, Gr(m, n))$ where

$$m = d + n(1 - g)$$

with respect to an appropriate linearization of the induced GL_m -action. The set of semistable points which arises from this construction will be a smooth quasi-projective variety and when n and d are coprime, its quotient will be the smooth projective variety $\mathcal{M}^C(n, d)$.

The construction of the moduli space $\mathcal{M}^C(n, d)$ from mapping spaces has the benefit of being closely related to the original Atiyah-Bott construction involving the classifying space of the gauge group. This follows from generalizations of Segal's work in [Seg79], which tell us that the inclusion of $Map_d(C, Gr(\ell, n))$ into the corresponding space $Map_d^{sm}(C, Gr(\ell, n))$ of smooth maps is a cohomology equivalence up to some degree tending to infinity with d . Very roughly speaking, the algebraic mapping spaces $Map_d(C, Gr(\ell, n))$, and the limiting space $Map_d(C, BGL_n)$, can be thought of as "algebraic approximations to Yang-Mills theory."

In order to study the motivic cohomology of $\mathcal{M}^C(n, d)$ following the last two sections, naïvely, we need to understand three things:

- the stabilizer groups of the action of GL_m on the open subscheme of semistable points in $Map_d(C, Gr(m, n))$ where $m = d + n(1 - g)$ (in particular, we would like all semistable points to be stable and hence their stabilizer groups to be finite),
- the instability stratification of $Map_d(C, Gr(m, n))$ and the equivariant motivic cohomologies of the unstable strata, and
- the GL_m -equivariant motivic cohomology of $Map_d(C, Gr(m, n))$.

Of course there are problems with each of these points. Firstly, the GL_m action on the subset of semistable points for the required linearization is not effective.

Regarding the third point, the space $\mathrm{Map}_d(C, \mathrm{Gr}(m, n))$ is not a smooth variety and thus the equivariant motivic cohomology is not defined, at least given the theory developed in §3. This also leads to problems with the second point, since while one can define an instability stratification, it will not necessarily have the properties advertised in Theorem 4.7 and thus Theorem 4.8 will not apply.

The first difficulty is the easiest to address: the centre $\mathbb{G}_{\mathbf{m}} \subset GL_m$ acts trivially on the space of maps and the quotient group PGL_m acts freely on the open subscheme of semistable points with quotient the moduli space $\mathcal{M}(n, d)$. In particular the stabilizer group at a semistable point, with respect to PGL_m , is trivial. Therefore, we may apply Lemma 3.7 and deduce that the integral motivic cohomology of $\mathcal{M}(n, d)$ is isomorphic to the PGL_n -equivariant motivic cohomology of the semistable points in $\mathrm{Map}_d(C, \mathrm{Gr}(m, n))$, under the appropriate hypotheses on m, n and d .

To address the second problem we replace $\mathrm{Map}_d(C, \mathrm{Gr}(m, n))$ with an open smooth subscheme $R_{n,d}$ (see §5.1) which admits $\mathcal{M}^C(n, d)$ as a GIT quotient of its semistable points. (Strictly speaking, we use the semistable points for a projective closure of $R_{n,d}$.) Thus one can define an instability stratification with properties much the same as those discussed in §4; however, one must be careful because $R_{n,d}$ is not proper.

For the third problem, understanding the GL_m -equivariant cohomology of $R_{n,d}$, we recall the construction of an auxiliary space given by Bifet, Ghione and Letizia (see [BGL94]), which is a scheme-theoretic version of Weil's original 1938 description of bundles on curves. They introduce ind-schemes of *matrix divisors*, which can be thought of as vector bundles on C equipped with trivializations at the generic point of C . The motivic cohomology of the relevant ind-scheme of matrix divisors is easy to compute via the Bialynicki-Birula decomposition (Theorem 4.3 above), since it is an inductive limit of smooth projective varieties over k : the motivic cohomology of spaces of matrix divisors can be reduced to studying motivic cohomology of symmetric products of curves (see §5.4 below).

In §5.3 we apply the procedure of §4 in detail to $R_{n,d}$, reducing the computation of the equivariant motivic cohomology of $R_{n,d}^{ss}$ (and equivalently, for n and d coprime, the motivic cohomology of $\mathcal{M}^C(n, d)$) to the equivariant motivic cohomology of $R_{n',d'}$ for various $n' \leq n$ and d' . In §5.5 we relate the equivariant motivic cohomology of $R_{n,d}$ to the motivic cohomology of a space of matrix divisors, which is itself computed in §5.4. Using this comparison, we describe the motivic cohomology of $\mathcal{M}^C(n, d)$ in terms of the motivic cohomology of symmetric powers of the curve C .

5.1 GIT construction of $\mathcal{M}(n, d)$

We will follow Newstead [New78]. Let us fix a smooth algebraic curve C of genus $g \geq 2$ over k an algebraically closed field. If L is a fixed degree λ line bundle on C , then tensoring by L gives an isomorphism

$$\varphi_L : \mathcal{M}(n, d) \xrightarrow{\sim} \mathcal{M}(n, d + n\lambda).$$

Consequently, in what follows we can assume that d is as large as we want.

Let $\mathcal{Q}_{m,n}$ denote the universal quotient bundle on $Gr(m,n)$. A morphism $f : C \rightarrow Gr(m,n)$ of degree d determines the rank n and degree d bundle $f^*\mathcal{Q}_{m,n}$ on C . Furthermore, such a morphism determines a surjection $\mathcal{O}_C^{\oplus m} \rightarrow f^*\mathcal{Q}_{m,n}$ via pull-back of the defining surjection $\mathcal{O}_{Gr(m,n)}^{\oplus m} \rightarrow \mathcal{Q}_{m,n}$. Thus, we have defined a map from the set of degree d morphisms $f : C \rightarrow Gr(m,n)$ to a set of rank n and degree d bundles over C equipped with a collection of m global generating sections, such that the GL_m -action on $Map_d(C, Gr(m,n))$ corresponds to changing the basis of generating sections of $f^*\mathcal{Q}_{m,n}$.

By taking d large enough, we may assume that any semistable bundle \mathcal{E} of degree d and rank n over C has the property that $H^1(C, \mathcal{E}) = 0$ and \mathcal{E} is generated by its sections. Then, by the Riemann-Roch theorem, $\dim H^0(C, \mathcal{E}) = d + n(1 - g)$, so set $m = d + n(1 - g)$ and define an open subscheme

$$R_{n,d} \subset Map_d(C, Gr(m,n)),$$

consisting of maps $f : C \rightarrow Gr(m,n)$ satisfying the following two conditions:

- i) the natural map $H^0(C, \mathcal{O}_C^{\oplus m}) \rightarrow H^0(C, f^*\mathcal{Q}_{m,n})$ is an isomorphism;
- ii) $H^1(C, f^*\mathcal{Q}_{m,n}) = 0$.

Let $R_{n,d}^s$ (respectively $R_{n,d}^{ss}$) denote the subset of $R_{n,d}$ consisting of maps f such that $f^*\mathcal{Q}_{m,n}$ is stable (respectively semistable).

Proposition 5.1 ([New78] §5) *For any pair n, d with d sufficiently large, the space $R_{n,d}$ is a smooth, quasi-projective scheme, on which GL_m acts naturally. The open subset $R_{n,d}^s$ (respectively $R_{n,d}^{ss}$) can be realized as the set of stable (respectively semistable) points for an appropriate linearization of the induced PGL_m -action on a projective completion of $R_{n,d}$. The group PGL_m acts freely on $R_{n,d}^s$ and the resulting quotient space $R_{n,d}^s/PGL_m$ is isomorphic to the moduli space $\mathcal{M}^C(n, d)$.*

In establishing this proposition, Newstead shows (see [New78] §5 and Remark 6.1) that if N is any sufficiently large integer then $R_{n,d}$ can be embedded as a nonsingular quasi-projective subvariety of the product $(Gr(m,n))^N$ via the map

$$f \mapsto (f(x_1), \dots, f(x_N)) \tag{17}$$

for suitable $x_1, \dots, x_N \in C$. Observe that PGL_m acts on $(Gr(m,n))^N$ diagonally. If d and N are chosen sufficiently large, then the locus of stable (respectively semistable) points in the closure $\overline{R_{n,d}}$ of $R_{n,d}$ in $(Gr(m,n))^N$, for an appropriate linearization of the PGL_m -action, coincides with the locus in $R_{n,d}$ representing stable (respectively semistable) bundles. Thus, the notation $R_{n,d}^s$ and $R_{n,d}^{ss}$ serves a convenient dual purpose. Moreover, for such d and N we have

$$\mathcal{M}^C(n, d) \cong R_{n,d}^{ss}/PGL_m = \overline{R_{n,d}}^{ss}/PGL_m.$$

Remark 5.2 *There is a natural evaluation morphism $ev : R_{n,d} \times C \longrightarrow Gr(m, n)$ which, at the level of points, sends a pair (f, x) corresponding to a map $f : C \longrightarrow Gr(m, n)$ and a point $x \in C$ to $f(x)$. The bundle $ev^* \mathcal{Q}_{m,n}$ has the universal property that for any map $f \in R_{n,d}$ the restriction of $ev^* \mathcal{Q}$ to C can be canonically identified with $f^* \mathcal{Q}$.*

5.2 Finite-dimensional approximations to Yang-Mills theory

For this subsection, assume that $k = \mathbb{C}$. As in §2.1 let us fix a complex vector bundle E of rank n and degree d , and denote by $\mathcal{C} = \mathcal{C}(n, d)$ the space of holomorphic structures on E . We have already remarked that for each degree d , there is a natural inclusion $Map_d(C, BGL_n) \hookrightarrow Map_d^{sm}(C, BGL_n)$ of algebraic maps into smooth maps, where the latter space is homotopy equivalent to the classifying space of the complexified gauge group $B\mathcal{G}_{\mathbb{C}}$.

There is a natural inclusion of Borel constructions $(R_{n,d})_{GL_m} = EGL_m \times_{GL_m} R_{n,d} \hookrightarrow B\mathcal{G}_{\mathbb{C}}$ defined as follows. Just as BGL_m is homotopy equivalent to the infinite Grassmannian $Gr(\infty, m)$, so we can identify EGL_m with the colimit over ℓ of the space of surjective linear maps from \mathbb{C}^ℓ to \mathbb{C}^m . Given a morphism $f : C \longrightarrow Gr(m, n)$ representing a point of $R_{n,d}$, and given a surjective linear map $e : \mathbb{C}^\ell \rightarrow \mathbb{C}^m$, we can define

$$F(f, e) : C \rightarrow Gr(\ell, m)$$

to be the morphism taking $x \in C$ to $f(x)$ thought of as an n -dimensional linear quotient of \mathbb{C}^ℓ rather than of \mathbb{C}^m by precomposition with e . Letting $\ell \rightarrow \infty$ this construction defines a morphism

$$(R_{n,d})_{GL_m} = EGL_m \times_{GL_m} R_{n,d} \rightarrow Map_d(C, BGL_n) \quad (18)$$

and it is shown in [Kir86a] (see Corollary 7.4 and Lemma 10.1) that the composition of this morphism with the inclusion of $Map_d(C, BGL_n)$ in $Map_d^{sm}(C, BGL_n) \simeq B\mathcal{G}_{\mathbb{C}}$ induces isomorphisms in cohomology up to some degree which tends to infinity as d tends to infinity for fixed n .

Remark 5.3 *This result is a limiting case of a generalization of Segal's theorem [Seg79] that the inclusion of the space of holomorphic maps $Map_d(C, \mathbb{P}^m)$ of degree d from a compact Riemann surface C to a projective space \mathbb{P}^m into the corresponding space $Map_d^{sm}(C, \mathbb{P}^m)$ of C^∞ maps from C to \mathbb{P}^m induces isomorphisms in cohomology up to degree $(d - 2g)(2m - 1) - 1$. Here $Map_d^{sm}(C, \mathbb{P}^m)$ is an infinite-dimensional space which is independent of d up to homotopy, whereas $Map_d(C, \mathbb{P}^m)$ is a finite-dimensional algebraic variety whose dimension tends to infinity with d . Segal's theorem tells us that, from the viewpoint of cohomology, as d tends to infinity the finite-dimensional varieties $Map_d(C, \mathbb{P}^m)$ are giving ever better approximations to the infinite-dimensional space $Map_d^{sm}(C, \mathbb{P}^m)$.*

Segal's result has been generalized to spaces of maps from C to, for example, flag manifolds and Grassmannians, and more recently [BHM01] to maps from

C to any compact Kähler manifold under a holomorphic action of a connected soluble Lie group S with an open orbit on which S acts freely.

The infinite-dimensional affine space \mathcal{C} is contractible, so the natural map from the Borel construction $\mathcal{C}_{\mathcal{G}_{\mathbb{C}}} = E\mathcal{G}_{\mathbb{C}} \times_{\mathcal{G}_{\mathbb{C}}} \mathcal{C}$ to $B\mathcal{G}_{\mathbb{C}}$ is a homotopy equivalence. Choosing a section gives us an inclusion

$$\mathrm{Map}_d(C, BGL_n) \hookrightarrow B\mathcal{G}_{\mathbb{C}} \hookrightarrow \mathcal{C}_{\mathcal{G}_{\mathbb{C}}};$$

we can think of this as given by a C^∞ identification of $f^* \mathcal{Q}_{m,n}$ with our fixed complex bundle E for each $f \in \mathrm{Map}_d(C, BGL_n)$ which provides a holomorphic structure on E . Composing this inclusion with (18) gives us an inclusion of Borel constructions

$$(R_{n,d})_{GL_m} \hookrightarrow (\mathcal{C})_{\mathcal{G}_{\mathbb{C}}} \quad (19)$$

which induces isomorphisms on cohomology up to arbitrarily high degree for d sufficiently large. If $\overline{\mathcal{G}_{\mathbb{C}}}$ denotes the quotient of $\mathcal{G}_{\mathbb{C}}$ by its central one-parameter subgroup consisting of automorphisms given by multiplication by nonzero-scalars, this discussion also shows that there is an induced map

$$i : (R_{n,d})_{PGL_m} \hookrightarrow (\mathcal{C})_{\overline{\mathcal{G}_{\mathbb{C}}}}, \quad (20)$$

again inducing isomorphisms on cohomology up to arbitrarily high degree for d large enough, which is compatible with the formation of quotients in the sense that the diagram

$$\begin{array}{ccc} (R_{n,d}^{ss})_{PGL_m} & \xrightarrow{i} & (\mathcal{C}^{ss})_{\overline{\mathcal{G}_{\mathbb{C}}}} \\ \downarrow & & \downarrow \\ R_{n,d} // PGL_m & \xrightarrow{\cong} & \mathcal{C} // \overline{\mathcal{G}_{\mathbb{C}}} \end{array} \quad (21)$$

commutes. If n and d are coprime, both terms in the bottom row are isomorphic to $\mathcal{M}(n, d)$.

While $R_{n,d}$ is not projective, we can still discuss the instability stratification associated with the linear action of PGL_m on $R_{n,d}$. Indeed, one can intersect the strata of the PGL_m -action on the singular space $\overline{R_{n,d}}$ (see §5.1) with $R_{n,d}$. In §2 we discussed the Yang-Mills stratification of \mathcal{C} . It is shown in [Kir86a] §11 that modulo subsets whose codimension tends to infinity as d tends to infinity, the inclusion of Borel constructions above takes the instability stratification of $R_{n,d}$ to the Yang-Mills stratification of $\mathcal{C}_{n,d}$. Moreover, although $R_{n,d}$ is not projective, its instability stratification is equivariantly perfect at least for cohomology up to some degree which tends to infinity with d , and the equivariant cohomology of its unstable strata can be described inductively in terms of the equivariant cohomology of $R_{\tilde{n}, \tilde{d}}^{ss}$ for varying $\tilde{n} < n$ and \tilde{d} , again up to some degree which tends to infinity with d . Thus the construction of $\mathcal{M}(n, d)$ as a finite-dimensional GIT quotient $R_{n,d} // PGL_m$ leads to an alternative derivation of the inductive formulae for calculating the Betti numbers of $\mathcal{M}(n, d)$ (for more details see [Kir86a]).

5.3 Equivariant motivic cohomology of strata

Assume again that k is an arbitrary algebraically closed field. Let us now study the action of PGL_m on $R_{n,d}$. Assume henceforth that n and d are coprime. Then PGL_m acts freely on $R_{n,d}^{ss}$ and by Lemma 3.7 the projection map $(R_{n,d}^{ss})_{PGL_m} \longrightarrow \mathcal{M}^C(n, d)$ induces isomorphisms in motivic cohomology:

$$H^{\bullet, \bullet}(\mathcal{M}^C(n, d), \mathbb{Z}) \xrightarrow{\sim} H_{PGL_m}^{\bullet, \bullet}(R_{n,d}^{ss}, \mathbb{Z}). \quad (22)$$

For a smooth variety X , the motivic cohomology group $H^{2,1}(X, \mathbb{Z})$ is canonically isomorphic to the Picard group. Thus, any line bundle L over X gives a class $c_{2,1}(L) \in H^{2,1}(X, \mathbb{Z})$ which we refer to as its $(2, 1)$ -chern class. In particular, let ξ denote $c_{2,1}(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^{2,1}(\mathbb{P}^n, \mathbb{Z})$. One can then compute the motivic cohomology ring of \mathbb{P}^n to be $H^{\bullet, \bullet}(\text{Spec } k, \mathbb{Z})[[\xi]]/\xi^{n+1}$, and there is an analogous theorem for projectivized vector bundles. Taking the appropriate limit, it follows that $H^{\bullet, \bullet}(B\mathbb{G}_m, \mathbb{Z}) \cong H^{\bullet, \bullet}(\text{Spec } k, \mathbb{Z})[[\xi]]$.

Now, the central one-parameter subgroup $\mathbb{G}_m \subset GL_m$ acts trivially on $R_{n,d}^{ss}$ and thus we have an \mathbb{A}^1 -weak equivalence

$$(R_{n,d}^{ss})_{GL_m} \xrightarrow{\sim} B\mathbb{G}_m \times (R_{n,d}^{ss})_{PGL_m}.$$

Applying the projective bundle formula, we obtain an isomorphism of rings:

$$H_{PGL_m}^{\bullet, \bullet}(R_{n,d}^{ss}, \mathbb{Z})[[\xi]] \xrightarrow{\sim} H_{GL_m}^{\bullet, \bullet}(R_{n,d}^{ss}, \mathbb{Z}). \quad (23)$$

We cannot apply the results of §4.4 directly to the instability stratification of $R_{n,d}$ because $R_{n,d}$ is not projective. However, motivic cohomology has the property that if Z is a subvariety of codimension c in a smooth variety X then

$$H^{i,j}(X - Z, \mathbb{Z}) \cong H^{i,j}(X, \mathbb{Z})$$

for weight $j < c$ (see Lemma 3.4). It therefore follows from a direct adaptation of the arguments of [Kir86a] §§8-13 that up to some weight which tends to infinity as $d \rightarrow \infty$ for fixed n the restriction map

$$H_{GL_m}^{\bullet, \bullet}(R_{n,d}, \mathbb{Z}) \longrightarrow H_{GL_m}^{\bullet, \bullet}(R_{n,d}^{ss}, \mathbb{Z}) \quad (24)$$

is surjective, and that Theorem 4.8 applies to the instability stratification of $R_{n,d}$. Moreover the instability stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $R_{n,d}$ is determined by Harder-Narasimhan type, modulo subvarieties whose codimension tends to infinity with d , in the following sense. Given any $M > 0$, if d is sufficiently large then to every Harder-Narasimhan type

$$\mu = (d_1/n_1, \dots, d_r/n_r)$$

as at (2), such that the codimension c_μ (given in Equation 4) of the corresponding Yang-Mills stratum \mathcal{C}_μ is at most M , we can attach an element $\beta(\mu)$ of the indexing set \mathcal{B} in such a way that

- i) if $\beta \in \mathcal{B}$ is not of the form $\beta(\mu)$ for some μ with $c_\mu \leq M$ then the corresponding stratum S_β of the instability stratification of $R_{n,d}$ has codimension greater than M ;
- ii) outside a subvariety of codimension at least M in $R_{n,d}$ we have for any μ with $c_\mu \leq M$ that $f \in S_{\beta(\mu)}$ if and only if $f^* \mathcal{Q}_{m,n}$ has Harder-Narasimhan type μ ;
- iii) $S_{\beta(\mu)}$ has codimension c_μ in $R_{n,d}$ and

$$S_{\beta(\mu)} \cong GL_m \times_{P_{\beta(\mu)}} Y_{\beta(\mu)}^{ss}$$

where $P_{\beta(\mu)}$ is a parabolic subgroup of GL_m with Levi subgroup

$$L_{\beta(\mu)} \cong \prod_{j=1}^r GL_{m_j}$$

for $m_j = d_j + n_j(1-g)$ where $\mu = (d_1/n_1, \dots, d_r/n_r)$, and $Y_{\beta(\mu)}^{ss}$ is smooth and is an $L_{\beta(\mu)}$ -equivariant Zariski locally trivial bundle with fibres isomorphic to affine spaces over

$$Z_{\beta(\mu)}^{ss} \cong \prod_{j=1}^r R_{n_j, d_j}^{ss}$$

modulo subvarieties of codimension at least M .

Thus, even though $R_{n,d}$ is not projective, nonetheless its instability stratification satisfies the analogues for motivic cohomology of the properties P, S1 and S2 described in §4.1. In particular, taking rational coefficients, the kernel of the surjection (24), for weights at most M , is, as a rational vector space, isomorphic to

$$\prod_{\mu \neq (d/n, \dots, d/n), c_\mu \leq M} H_{GL_m}^{\bullet-2c_\mu, \bullet-c_\mu}(S_{\beta(\mu)}, \mathbb{Q})$$

with

$$H_{GL_m}^{\bullet, \bullet}(S_{\beta(\mu)}, \mathbb{Q}) \cong H_{\prod_{j=1}^r GL_{d_j + n_j(1-g)}}^{\bullet, \bullet} \left(\prod_{j=1}^r R_{n_j, d_j}^{ss}, \mathbb{Q} \right). \quad (25)$$

Remark 5.4 *Motivic cohomology does not have a Künneth formula in the sense of having a convergent Künneth spectral sequence for general smooth schemes (see [DI05] for more information about Künneth spectral sequences in this context). Therefore formula (25) is not as simple as its topological counterpart (2.1.3).*

Remark 5.5 *To justify this argument, which involves an application of the methods of Theorem 4.8, we need to check that the action of GL_m on $R_{n,d}$ is manageable, in the sense of Theorem 4.4. For this, let us first describe the tangent space to $R_{n,d}$ at a point. Let $\mathcal{S}_{m,n}$ denote the universal subbundle of*

the trivial rank m -bundle on $Gr(m, n)$; as before $\mathcal{Q}_{m, n}$ denotes the universal quotient bundle. Consider a closed point of $R_{n, d}$ defined by $f : C \rightarrow Gr(m, n)$, and observe that the Zariski tangent space to $R_{n, d}$ at this point is canonically isomorphic to $H^0(C, f^*(\mathcal{S}_{m, n}^\vee \otimes \mathcal{Q}_{m, n}))$.

Suppose that the Harder-Narasimhan filtration of $f^*\mathcal{Q}_{m, n}$ takes the form:

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = f^*\mathcal{Q}_{m, n},$$

with type specified by

$$\mu = (d_1/n_1, \dots, d_r/n_r).$$

Then (away from a subvariety of $R_{n, d}$ whose codimension tends to infinity with d) f belongs to $Y_{\beta(\mu)}^{ss}$ if and only if the linear subspaces

$$0 = H^0(C, \mathcal{E}_0) \subset H^0(C, \mathcal{E}_1) \subset \cdots \subset H^0(C, \mathcal{E}_j) \subset \cdots \subset H^0(C, \mathcal{E}_r) = k^m$$

are spanned by the subsets $\{e_1, \dots, e_{m_j}\}$ of the standard basis $\{e_1, \dots, e_m\}$ of k^m . Moreover $f \in Z_{\beta(\mu)}^{ss}$ if and only if in addition each \mathcal{E}_j is a direct sum

$$\mathcal{E}_j = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_j$$

of semistable bundles \mathcal{F}_j of rank n_j and degree d_j where $H^0(C, \mathcal{F}_j) \subseteq k^m$ is spanned by the subset $\{e_{m_1+\dots+m_{j-1}+1}, \dots, e_{m_1+\dots+m_j}\}$ of the standard basis of k^m . Equivalently $f : C \rightarrow Gr(m, n)$ is given by the composition of a map of the form

$$(f_1, \dots, f_r) : C \rightarrow \prod_{j=1}^r Gr(m_j, n_j)$$

(for some $f_j \in R_{n_j, d_j}^{ss}$) with the standard embedding of $\prod_{j=1}^r Gr(m_j, n_j)$ in $Gr(m, n)$, where the restrictions of the universal subbundle $\mathcal{S}_{m, n}$ and quotient bundle $\mathcal{Q}_{m, n}$ on $Gr(m, n)$ to $\prod_{j=1}^r Gr(m_j, n_j)$ are given by

$$\bigoplus_{j=1}^r \mathcal{S}_{m_j, n_j} \quad \text{and} \quad \bigoplus_{j=1}^r \mathcal{Q}_{m_j, n_j}.$$

Then $f_j^*(\mathcal{Q}_{m_j, n_j}) \cong \mathcal{F}_j$ is semistable of rank n_j and degree d_j and the tangent space to $R_{n, d}$ at f is

$$H^0(C, f^*(\mathcal{S}_{m, n}^\vee \otimes \mathcal{Q}_{m, n})) \cong \bigoplus_{i, j} H^0(C, f_i^*(\mathcal{S}_{m_i, n_i}^\vee) \otimes f_j^*(\mathcal{Q}_{m_j, n_j})),$$

while the tangent space to $Y_{\beta(\mu)}^{ss}$ at f is

$$\bigoplus_{i \leq j} H^0(C, f_i^*(\mathcal{S}_{m_i, n_i}^\vee) \otimes f_j^*(\mathcal{Q}_{m_j, n_j})).$$

If $\xi \in LieGL_m = \bigoplus_{i, j} (k^{m_i})^\vee \otimes k^{m_j}$ has decomposition $\xi = (\xi_{i, j})$ with $\xi_{i, j} \in (k^{m_i})^\vee \otimes k^{m_j}$ then the infinitesimal action of ξ at f is given by

$$\xi_f = (\xi_{i, j}^f) \in \bigoplus_{i, j} H^0(C, f_i^*(\mathcal{S}_{m_i, n_i}^\vee) \otimes f_j^*(\mathcal{Q}_{m_j, n_j}))$$

where $\xi_{i,j}^f \in H^0(C, f_i^*(S_{m_i, n_i}^\vee) \otimes f_j^*(Q_{m_j, n_j}))$ is the image of $\xi_{i,j}$ under the map

$$(k^{m_i})^\vee \otimes k^{m_j} \rightarrow H^0(C, f_i^*(S_{m_i, n_i}^\vee) \otimes f_j^*(Q_{m_j, n_j})) \quad (26)$$

which comes from the bundle surjection

$$(k^{m_i})^\vee \otimes k^{m_j} \otimes \mathcal{O}_C \rightarrow f_i^*(S_{m_i, n_i}^\vee) \otimes f_j^*(Q_{m_j, n_j})$$

which factors through $(k^{m_i})^\vee \otimes f_j^*(Q_{m_j, n_j})$. Since $H^0(C, (k^{m_i})^\vee \otimes f_j^*(Q_{m_j, n_j}))$ is isomorphic to $(k^{m_i})^\vee \otimes H^0(C, f_j^*(Q_{m_j, n_j})) \cong (k^{m_i})^\vee \otimes k^{m_j}$ and the bundle $f_i^*(Q_{m_i, n_i}^\vee) \otimes f_j^*(Q_{m_j, n_j}) \cong \mathcal{F}_j^\vee \otimes \mathcal{F}_i$ has no global sections when $i > j$ (as \mathcal{F}_i and \mathcal{F}_j are both semistable with $\mu(\mathcal{F}_j) > \mu(\mathcal{F}_i)$), it follows from the long exact sequence

$$\rightarrow H^0(C, f_i^*(Q_{m_i, n_i}^\vee) \otimes f_j^*(Q_{m_j, n_j})) \rightarrow H^0(C, (k^{m_i})^\vee \otimes f_j^*(Q_{m_j, n_j})) \rightarrow H^0(C, f_i^*(S_{m_i, n_i}^\vee) \otimes f_j^*(Q_{m_j, n_j})) \rightarrow$$

that the map (26) is injective when $i > j$. Hence the action of GL_m on $R_{n,d}$ is manageable, at least away from a subvariety of $R_{n,d}$ whose codimension tends to infinity with d .

5.4 Matrix divisors and adèles

In order to complete our inductive procedure for understanding the motivic cohomology of the moduli spaces $\mathcal{M}^C(n, d)$, what remains is to compute the equivariant motivic cohomology $H_{GL_m}^{\bullet, \bullet}(R_{n,d}, \mathbb{Z})$. Taking $k = \mathbb{C}$ and using ordinary cohomology, a method for doing this was sketched in §5.2, but this argument relied on Atiyah and Bott's computation of the cohomology of the classifying space of the complexified gauge group together with a generalization of Segal's theorem [Seg79] and must be replaced by something suitably algebraic for motivic cohomology. To achieve this, we will relate the algebraic approximations to Yang-Mills theory to a dual description in terms of matrix divisors.

A *matrix divisor* of rank n on a smooth projective curve C is a locally free subsheaf of the sheaf $k(C)^{\oplus n}$ (where $K = k(C)$ is the function field of C). Given a vector bundle \mathcal{F} , observe that specifying an inclusion $\mathcal{F} \hookrightarrow k(C)^{\oplus n}$ is equivalent to specifying a trivialization of \mathcal{F} at the generic point.

Remark 5.6 *Matrix divisors have the following interpretation. If K is the function field $k(C)$ of C , we can define \mathbb{A}_K in a manner analogous to that for finite fields (see §2.2 above). For every k -point x of C , set $\hat{\mathcal{O}}_x$ and \hat{K}_x to be the completed local ring and its field of fractions respectively. For any finite set $S \subset C(k)$, define $\mathbb{A}_S = \prod_{x \in S} \hat{K}_x \times \prod_{x \in C(k) - S} \hat{\mathcal{O}}_x$. Then set \mathbb{A}_K to be the colimit of \mathbb{A}_S as S varies through the partially ordered set of subsets of $C(k)$. The set $\text{Bun}_{GL_n}(k)$ of isomorphism classes of vector bundles of rank n and degree d on C can be identified with elements of the double coset space $\mathfrak{K} \backslash GL_n(\mathbb{A}_K) / GL_n(K)$, where \mathfrak{K} is a subgroup which when $d = 0$ is $\prod_{x \in C(k)} \hat{\mathcal{O}}_x$. The elements of $GL_n(\mathbb{A}_K)$ can be identified with the space of collections $(V, \varphi_\eta, \{\varphi_x\})$ consisting of a vector bundle V on C , a trivialization at the generic point, and a trivialization along the formal disc at every point $x \in C(k)$. Thus matrix divisors can be identified with elements of the coset space $\mathfrak{K} \backslash GL_n(\mathbb{A}_K)$.*

For a fixed effective divisor D , we can consider the set of locally free rank n \mathcal{O}_C -submodules \mathcal{F} of $k(C)^{\oplus n}$ which are contained in $\mathcal{O}_C(D)^{\oplus n}$. The set of such embeddings in fact forms the set of k -points of a disconnected smooth projective variety whose components are indexed by the degree d of \mathcal{F} . Following [BGL94], we denote these components by $\text{Div}_{C/k}^{n,d}(D)$. If $D \leq D'$, then we have a closed embedding

$$\text{Div}_{C/k}^{n,d}(D) \hookrightarrow \text{Div}_{C/k}^{n,d}(D'). \quad (27)$$

We set $\text{Div}_{C/k}^{n,d} = \text{colim}_D \text{Div}_{C/k}^{n,d}(D)$; then $\text{Div}_{C/k}^{n,d}$ is a ind-smooth projective variety.

Motivic cohomology of the space of matrix divisors

The motivic cohomology of the space $\text{Div}_{C/k}^{n,d}(D)$ can be studied using the Bialynicki-Birula decomposition associated with a generic one-parameter subgroup \mathbb{G}_m of a maximal torus in GL_n (see Theorem 4.4).

Lemma 5.7 *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a vector of non-negative integers, let $c_{\mathbf{d}} = \sum_{1 \leq i \leq n} (i-1)d_i$, and write $|\mathbf{d}| = d_1 + \dots + d_n$. Then*

$$H^{\bullet, \bullet}(\text{Div}_{C/k}^{n,d}(D)) \cong \bigoplus_{|\mathbf{d}|=d} H^{\bullet-2c_{\mathbf{d}}, \bullet-c_{\mathbf{d}}}(C^{(d_1)} \times \dots \times C^{(d_n)}) \quad (28)$$

where $C^{(j)}$ is the j th symmetric power of the curve C .

The proof is essentially that given in [Bif89, BGL94]. The automorphism group of the bundle $\mathcal{O}_C(D)$ is \mathbb{G}_m . Thus we obtain an action of the split maximal torus $T \subset GL_n$ of diagonal matrices on the sum $\mathcal{O}_C(D)^{\oplus n}$ on $\text{Div}_{C/k}^{n,d}(D)$, which can be thought of as a torus in $GL_n(k(C))$. The components of the fixed-point set of this torus action correspond to matrix divisors which are direct sums of line bundles of the form

$$\mathcal{O}_C(D_1) \oplus \dots \oplus \mathcal{O}_C(D_n)$$

and, by taking the cokernels of the inclusions of such bundles into $\mathcal{O}_C(D)^{\oplus n}$, give rise to torsion sheaves on C . Thus we obtain an identification of the fixed-point loci with products of Hilbert schemes of points on C , or equivalently (since C is one-dimensional) symmetric powers of C .

Remark 5.8 *The ℓ -adic cohomology of $\text{Div}_{C/k}^{n,d}(D)$ is calculated in [Bif89] (see also [BGL94] Proposition 4.2) using the computation above together with Macdonald's computation of the generating function of the cohomology of symmetric products of a curve.*

Remark 5.9 *Note that when j is large enough the symmetric power $C^{(j)}$ of C is a projective bundle over the abelian variety $\mathcal{M}^C(1, j)$ which is independent of j up to isomorphism. Thus if d_1 is large enough the product $C^{(d_1)} \times \dots \times C^{(d_n)}$*

is a projective bundle over $\mathcal{M}^C(1, 0) \times C^{(d_2)} \times \dots \times C^{(d_n)}$ and hence its motivic cohomology is independent of d_1 up to some level which tends to infinity with d_1 . Note also that $c_{\mathbf{d}}$ as defined in Lemma 5.7 for $\mathbf{d} = (d_1, \dots, d_n)$ is independent of d_1 . From this and (28) it follows that the motivic cohomology of $\text{Div}_{C/k}^{n,d}(D)$ stabilizes as the degree of D tends to infinity.

5.5 Linking maps to Grassmannians with matrix divisors

Given any morphism $f : C \rightarrow \text{Gr}(m, n)$ representing a point of $R_{n,d}$, we obtain a surjective morphism $\mathcal{O}_C^{\oplus m} \rightarrow f^* \mathcal{Q}_{m,n}$. Dualizing this morphism gives an injective map $f^* \mathcal{Q}_{m,n}^\vee \hookrightarrow \mathcal{O}_C^{\oplus m}$ of \mathcal{O}_C -modules. Choosing a morphism $\mathcal{O}_C^{\oplus m} \rightarrow \mathcal{O}_C^{\oplus n}$ such that the composite map $f^* \mathcal{Q}_{m,n}^\vee \rightarrow \mathcal{O}_C^{\oplus n}$ is injective thus gives rise, by definition, to a matrix divisor.

Recall that in order to complete our inductive procedure for studying the motivic cohomology of the moduli spaces $\mathcal{M}^C(n, d)$ it remains to compute the equivariant motivic cohomology $H_{GL_m}^{\bullet, \bullet}(R_{n,d}, \mathbb{Z})$, or equivalently the motivic cohomology of the Borel construction $(R_{n,d})_{GL_m}$ associated with the action of GL_m on $R_{n,d}$. It follows from the discussion in §5.2 that we can identify $(R_{n,d})_{GL_m}$ with the image of the morphism

$$(R_{n,d})_{GL_m} \rightarrow \text{Map}_d(C, BGL_n) = \text{colim}_\ell \text{Map}_d(C, \text{Gr}(\ell, n))$$

given at (5.2.1). Thus $(R_{n,d})_{GL_m}$ is the colimit

$$\text{colim}_\ell (R_{n,d})_{GL_m}^\ell$$

where $(R_{n,d})_{GL_m}^\ell$ is the open subscheme of $\text{Map}_d(C, \text{Gr}(\ell, n))$ consisting of maps $F : C \rightarrow \text{Gr}(\ell, n)$ satisfying

- (i) the map of sections $H^0(\alpha_F) : k^\ell = H^0(C, \mathcal{O}_C^{\oplus \ell}) \rightarrow H^0(C, F^*(\mathcal{Q}_{\ell,n}))$ associated with the natural map $\alpha_F : \mathcal{O}_C^{\oplus \ell} \rightarrow F^*(\mathcal{Q}_{\ell,n})$ is surjective, and
- (ii) $H^1(C, F^*(\mathcal{Q}_{\ell,n})) = 0$.

We will complete our inductive procedure by showing that if d is chosen suitably and ℓ and D are sufficiently large then we have canonical isomorphisms

$$H^{i,j}((R_{n,d})_{GL_m}^\ell, \mathbb{Z}) \cong H^{i,j}(\text{Div}_{C/k}^{n,\tilde{d}}(D), \mathbb{Z})$$

where $\tilde{d} = n \deg D - d$, for all j up to some weight which can be taken to be arbitrarily high. We will do this by comparing both $(R_{n,d})_{GL_m}^\ell$ and $\text{Div}_{C/k}^{n,\tilde{d}}(D)$ with an auxiliary scheme $U_{n,d}^\ell$ which is an open subscheme of the product

$$\text{Map}_d(C, \text{Gr}(\ell, n)) \times ((k^\ell)^\vee \otimes k^n)$$

where we will interpret elements $\varphi \in k^{\ell n} = ((k^\ell)^\vee \otimes k^n)$ as linear maps from the standard ℓ -dimensional vector space k^ℓ to the n -dimensional vector space k^n . Given any such linear map φ and a scheme S , we let $\varphi_S : k^\ell \otimes \mathcal{O}_S \rightarrow k^n \otimes \mathcal{O}_S$ denote the corresponding morphism of trivial vector bundles over S .

Definition 5.10 Let $U_{n,d}^\ell$ be the open subscheme of $\text{Map}_d(C, \text{Gr}(\ell, n)) \times ((k^\ell)^\vee \otimes k^n)$ consisting of pairs (F, φ) where $F : C \rightarrow \text{Gr}(\ell, n)$ and $\varphi : k^\ell \rightarrow k^n$ satisfy (i) and (ii) above and also (iii) the composition of $\varphi_C : \mathcal{O}_C^{\oplus \ell} \rightarrow \mathcal{O}_C^{\oplus n}$ with the dual of $\alpha_F : \mathcal{O}_C^{\oplus \ell} \rightarrow F^*(\mathcal{Q}_{\ell,n}^\vee)$ is an injective map of \mathcal{O}_C -modules $F^*(\mathcal{Q}_{\ell,n}^\vee) \rightarrow \mathcal{O}_C^{\oplus n}$.

Let $\tilde{d} = n \deg D - d$. Then we have morphisms

$$\hat{\theta} : U_{n,d}^\ell \rightarrow (R_{n,d})_{GL_m}^\ell \quad \text{and} \quad \tilde{\theta} : U_{n,d}^\ell \rightarrow \text{Div}_{C/k}^{n,\tilde{d}}(D)$$

such that $\hat{\theta}(F, \varphi) = F$ while $\tilde{\theta}$ sends (F, φ) to the matrix divisor

$$F^*(\mathcal{Q}_{\ell,n}^\vee) \otimes \mathcal{O}_C(D) \rightarrow \mathcal{O}_C(D)^{\oplus n}$$

obtained by tensoring the injective map of \mathcal{O}_C -modules $F^*(\mathcal{Q}_{\ell,n}^\vee) \rightarrow \mathcal{O}_C^{\oplus n}$, which is the composition of $\varphi_C : \mathcal{O}_C^{\oplus \ell} \rightarrow \mathcal{O}_C^{\oplus n}$ with the dual of $\alpha_F : \mathcal{O}_C^{\oplus \ell} \rightarrow F^*(\mathcal{Q}_{\ell,n}^\vee)$, with the identity on $\mathcal{O}_C(D)$.

The fibre of $\hat{\theta} : U_{n,d}^\ell \rightarrow (R_{n,d})_{GL_m}^\ell$ over $F : C \rightarrow \text{Gr}(\ell, n)$ consists of those $\varphi : k^\ell \rightarrow k^n$ such that the composition of $\varphi_C : \mathcal{O}_C^{\oplus \ell} \rightarrow \mathcal{O}_C^{\oplus n}$ with the dual of $\alpha_F : \mathcal{O}_C^{\oplus \ell} \rightarrow F^*(\mathcal{Q}_{\ell,n}^\vee)$ is an injective map of \mathcal{O}_C -modules $F^*(\mathcal{Q}_{\ell,n}^\vee) \rightarrow \mathcal{O}_C^{\oplus n}$. The codimension in $(k^\ell)^\vee \otimes k^n$ of the complement of any such fibre tends to infinity as $\ell \rightarrow \infty$ so we have:

Lemma 5.11 *The morphism $\hat{\theta} : U_{n,d}^\ell \rightarrow (R_{n,d})_{GL_m}^\ell$ induces isomorphisms of motivic cohomology up to a weight which tends to infinity with ℓ .*

Suppose now that \mathcal{F} is a locally free rank n \mathcal{O}_C -submodule of $\mathcal{O}_C(D)^{\oplus n}$ representing a point of $\text{Div}_{C/k}^{n,\tilde{d}}(D)$ and satisfying the two conditions:

- i) $E^\vee \otimes \mathcal{O}_C(D)$ is generated by sections, and
- ii) $H^1(C, E^\vee \otimes \mathcal{O}_C(D))$ vanishes.

Then the fibre of $\tilde{\theta} : U_{n,d}^\ell \rightarrow \text{Div}_{C/k}^{n,\tilde{d}}(D)$ at this point consists of all pairs (φ, ψ) where $\varphi^\vee : k^{\oplus n} \rightarrow k^{\oplus \ell}$ is injective and $\psi : k^\ell \rightarrow H^0(C, \mathcal{F}^\vee(D))$ is surjective and $\psi \circ \varphi : k^{\oplus n} \rightarrow H^0(C, \mathcal{O}_C^{\oplus n})$ is the map on sections induced by the dual of $E(-D) \hookrightarrow \mathcal{O}_C^{\oplus n}$. This fibre is therefore an affine bundle over the space of injective linear maps from k^n to k^ℓ , which is an open subscheme of $(k^n)^\vee \otimes k^\ell$ whose complement has codimension tending to infinity with ℓ .

Note that the complement in $\text{Div}_{C/k}^{n,\tilde{d}}(D)$ of the open subscheme where these conditions (i) and (ii) above are satisfied has codimension tending to infinity with $\deg(D)$. Putting these together yields

Lemma 5.12 *If ℓ and $\deg(D)$ are sufficiently large then $\tilde{\theta} : U_{n,d}^\ell \rightarrow \text{Div}_{C/k}^{n,\tilde{d}}(D)$ induces isomorphisms of motivic cohomology up to arbitrarily high weights.*

Combining Lemmas 5.11 and 5.12 we obtain:

Corollary 5.13 *Given any $M > 0$, if d is sufficiently large and if ℓ and $\deg(D)$ are large enough (depending on d), we have canonical isomorphisms in motivic cohomology*

$$H^{i,j}(\mathrm{Div}_{C/k}^{n,\tilde{d}}(D); \mathbb{Z}) \cong H_{GL_m}^{i,j}(R_{n,d}; \mathbb{Z}) \quad (29)$$

for $j < M$.

Remark 5.14 *We can think of $(R_{n,d})_{GL_m}^\ell$ mapping to the moduli stack $\mathcal{M}^C(n, d)$ of bundles over C of rank n and degree d , by associating with $F : C \rightarrow \mathrm{Gr}(\ell, n)$ the bundle $F^*(\mathcal{Q}_{\ell,n})$, with ‘fibre’ $EGL_m/\mathbb{G}_m = \mathbb{P}^\infty$. Similarly we can think of $\mathrm{Div}_{C/k}^{n,\tilde{d}}(D)$ mapping to the moduli stack $\mathcal{M}^C(n, \tilde{d})$ where $\tilde{d} = n \deg(D) - d$ by associating with the matrix divisor $\mathcal{F} \hookrightarrow \mathcal{O}_C(D)^{\oplus n}$ the bundle $E^\vee \otimes \mathcal{O}(D)$. The compositions of these with $\hat{\theta} : U_{n,d}^\ell \rightarrow (R_{n,d})_{GL_m}^\ell$ and $\tilde{\theta} : U_{n,d}^\ell \rightarrow \mathrm{Div}_{C/k}^{n,\tilde{d}}(D)$ respectively agree modulo the isomorphism from $\mathcal{M}^C(n, d)$ to $\mathcal{M}^C(n, n \deg D - d)$ given by $\mathcal{F} \rightarrow \mathcal{F}^\vee \otimes \mathcal{O}(D)$.*

Remark 5.15 *Associating with a matrix divisor $\mathcal{F} \hookrightarrow k(C)^{\oplus n}$ the bundle \mathcal{F} and forgetting the embedding of \mathcal{F} in $k(C)^{\oplus n}$ determines a map from the space of matrix divisors on C to the moduli stack of rank n bundles on C ; we refer to this map as the Abel-Jacobi map θ . When n and d are coprime, it is shown in [BGL94] that if we denote by $(\mathrm{Div}_{C/k}^{n,d})^{ss}$ the space consisting of matrix divisors whose underlying locally free sheaves are semistable, the Abel-Jacobi map restricts to a morphism $\theta : (\mathrm{Div}_{C/k}^{n,d})^{ss} \rightarrow \mathcal{M}^C(n, d)$. The isomorphism (29) is essentially equivalent to Theorem 4.5 of [Dhi06], which says that the Abel-Jacobi map from the ind-variety $\mathrm{Div}_{C/k}^{n,d}$ to the moduli stack of rank n and degree d vector bundles on C is a quasi-isomorphism (i.e. induces isomorphisms on cohomology groups).*

The space $(\mathrm{Div}_{C/k}^{n,d})^{ss}$ can be identified as an open subset of a certain projective bundle over $\mathcal{M}^C(n, d)$. More precisely, $(\mathrm{Div}_{C/k}^{n,d})^{ss}$ is constructed as an ind-scheme by defining an inductive system of schemes $\mathrm{Div}_{C/k}^{n,d}(D)^{ss}$ indexed by effective divisors D on C . The fibre of $\mathrm{Div}_{C/k}^{n,d}(D)^{ss}$ at a semistable vector bundle \mathcal{E} can be identified with the subset of $\mathbb{P}(H^0(C, \mathcal{E}^\vee \otimes \mathcal{O}_C(D)^{\oplus n}))$ corresponding to injective \mathcal{O}_C -module maps $\mathcal{E} \hookrightarrow \mathcal{O}_C(D)^{\oplus n}$. By [BGL94] Lemma 8.2, the codimension of the complement of the fibre of $\mathrm{Div}_{C/k}^{n,d}(D)^{ss}$ over \mathcal{E} in $\mathbb{P}(H^0(C, \mathcal{E}^\vee \otimes \mathcal{O}_C(D)^{\oplus n}))$ tends to infinity as $\deg(D) \rightarrow \infty$. It then follows from the projective bundle theorem for motivic cohomology that the motivic cohomology of $(\mathrm{Div}_{C/k}^{n,d})^{ss}$ is given by

$$H^{\bullet,\bullet}((\mathrm{Div}_{C/k}^{n,d})^{ss}) \cong H^{\bullet,\bullet}(\mathcal{M}^C(n, d))[[\xi]]$$

where ξ corresponds to the (2,1) Chern class of the projective bundle described above over $\mathrm{Div}_{C/k}^{n,d}(D)^{ss}$ for suitably large D .

We can stratify $\mathrm{Div}_{C/k}^{n,d}(D)$ according to the Harder-Narasimhan type (2) of the underlying bundle \mathcal{F} ; this is how Bifet, Ghione and Letizia obtained their inductive formulae for calculating the Betti numbers of $\mathcal{M}(n,d)$. Equivalently the ind-variety $\mathrm{Div}_{C/k}^{n,d}$ can be stratified by Harder-Narasimhan type; the resulting strata are called Shatz strata in [BGL94]. The Shatz stratification is perfect and the cohomology of the strata can be described inductively in a manner identical to our discussion for $R_{n,d}$ but in terms of products of spaces of the form $\mathrm{Div}_{C/k}^{n_j,d_j}(D)^{ss}$ with $0 < n_j < n$. If $D \leq D'$, the closed immersion $\mathrm{Div}_{C/k}^{n,d}(D) \hookrightarrow \mathrm{Div}_{C/k}^{n,d}(D')$ is a stratified morphism, and the limit of these finite dimensional approximating strata as $\deg(D) \rightarrow \infty$ agrees with the Shatz stratification.

Building on the work of Bifet, Ghione and Letizia, del Baño [dB01] showed, using the same Shatz stratification of $\mathrm{Div}_{C/k}^{n,d}$ but now working over a characteristic 0 not necessarily algebraically closed field, that one can in the same manner compute the rational Chow motive of the space of matrix divisors, and thence the virtual Chow motive of the moduli space of stable bundles on a curve C . To do this he needed to understand the Chow motive of symmetric products of C ; he therefore established a motivic version of MacDonald's formulae for symmetric products of C [dB01, Section 3.3]. He also needed to show that the Bialynicki-Birula stratification of a smooth projective variety induces a direct sum decomposition of the corresponding rational Chow motive; this result was fundamental to Brosnan's more general theorems on Bialynicki-Birula decompositions [Bro05]. He thereby showed that the Chow motive of $\mathcal{M}(n,d)$ lies in the category generated by the motive of the curve and provided a formula for the "virtual motive" of $\mathcal{M}^C(n,d)$, with a closed form expression for the motivic Poincaré polynomial.

When n and d are coprime, combining (22), (23), (24), (25), (29) and (28) with the application of Theorem 4.8 to the instability stratification of $R_{n,d}$ as in §5.3 leads to an inductive method for calculating the motivic cohomology groups $H^{\bullet,\bullet}(\mathcal{M}(n,d), \mathbb{Q})$ in terms of the motivic cohomology of products of Jacobians of C and the closely related Hilbert schemes of points on C .

Remark 5.16 *Similar arguments apply to moduli spaces of parabolic bundles over C (cf. [BR96, Hol00, HJ00, MS80, Nit86, Nit96, Nit97]).*

Remark 5.17 *When n and d have a common factor then (22) is no longer valid, but (23), (24), (25), (29) and (28) still provide an inductive procedure for calculating the equivariant motivic cohomology groups $H_{GL_m}^{\bullet,\bullet}(R^{ss}, \mathbb{Q})$ of $R_{n,d}^{ss}$ or equivalently the groups $H_{PGL_m}^{\bullet,\bullet}(R^{ss}, \mathbb{Q})$. The codimension of the complement of R^s in R^{ss} is at least $(g-1)(n-1)$ (cf. e.g. [Kir86b] §3 or [Dhi06] Cor 5.5) and hence if $\mathcal{M}^s(n,d)$ denotes the moduli space of stable bundles we have*

$$H_{PGL_m}^{i,j}(R^{ss}, \mathbb{Q}) \cong H_{PGL_m}^{i,j}(R^s, \mathbb{Q}) \cong H^{i,j}(\mathcal{M}^s(n,d), \mathbb{Q})$$

for $j < (g-1)(n-1)$ by Remark 4.7, giving us the motivic cohomology of the moduli space $\mathcal{M}^s(n,d)$ in low degrees. We can also use the method described

in §4.4 above to study the motivic cohomology of a partial desingularization $\tilde{\mathcal{M}}(n, d)$ of $\mathcal{M}(n, d)$ when k has characteristic zero (cf. [Kir86b]).

6 Conclusion

We have described three equivalent inductive procedures leading to calculation of the Betti numbers of the moduli spaces $\mathcal{M}(n, d)$ when n and d are coprime. The three procedures all rely on stratifications linked to the Harder-Narasimhan type of a holomorphic vector bundle, but they differ technically.

(i) The approach of Harder and Narasimhan [HN75] and Desale and Ramanan [DR75] via the Weil conjectures uses Tamagawa measures and reduces to the fact that the Tamagawa number τ_{SL_n} of SL_n is 1.

(ii) The approach of Atiyah and Bott [AB83] via Yang-Mills theory uses equivariant Morse theory and reduces to a simple description of the cohomology of the classifying space of the gauge group.

(iii) The moduli space $\mathcal{M}(n, d)$ can be expressed as a GIT quotient $R_{n,d}/PGL_m$ and both versions (using equivariant Morse theory and counting points of an associated variety over finite fields) of the procedure described in §4.1 for calculating Betti numbers of GIT quotients $X//G$ can be applied (though extra care is needed since $R_{n,d}$ is not projective); the GL_m -equivariant cohomology of $R_{n,d}$ can be expressed in terms of the cohomology of symmetric products of the curve C , by relating the Borel construction $(R_{n,d})_{GL_m}$ to a space of matrix divisors.

We have seen in §5 that the third of these approaches can be made to work for motivic cohomology; it also provides a link between the other two approaches. To link the third approach with the approach of Atiyah and Bott via Yang-Mills theory, we consider the inclusion

$$EGL_m \times_{GL_m} R_{n,d} \longrightarrow \text{Map}_d(C, BSL_n) \longrightarrow \text{Map}_d^{sm}(C, BSL_n) = B\mathcal{G}_C$$

defined as at (5.2.1). By a generalization of Segal's theorem on the topology of spaces of rational maps [Seg79] (see Remark 5.3) this induces isomorphisms

$$H^j(B\mathcal{G}_C) \cong H_{GL_m}^j(R_{n,d})$$

for d sufficiently large, which gives us a direct link between approaches (ii) and (iii) (see [Kir86a] for more details).

To link the third approach with the first using Tamagawa measures, we consider the variety $U_{n,d}^\ell$ as in Definition 5.10. The argument given in §5.5, using the maps $\theta : U_{n,d}^\ell \rightarrow (R_{n,d})_{GL_m}^\ell$ and $\tilde{\theta} : U_{n,d}^\ell \rightarrow \text{Div}_{C/k}^{n,\tilde{d}}(D)$ to the space of matrix divisors $\text{Div}_{C/k}^{n,\tilde{d}}(D)$ to show that the GL_m -equivariant motivic cohomology of $R_{n,d}$ is isomorphic to the motivic cohomology of $\text{Div}_{C/k}^{n,\tilde{d}}(D)$ (up to some weight which can be made arbitrarily large), also allows us to relate the number of points in corresponding varieties defined over finite fields. Interpreting matrix divisors in terms of adèles as in Remark 5.6 then gives us a direct link between

the ‘counting points’ version of approach (iii) (see §4.1) and the Tamagawa measures used in approach (i). It also provides an alternative method for proving that the Tamagawa number of SL_n is 1 by using the Bialynicki-Birula stratification of $\mathrm{Div}_{C/k}^{n,\bar{d}}(D)$ and counting points on symmetric products of associated curves over finite fields (cf. [BDb]).

In conclusion, both the arithmetic approach (i) and the Yang-Mills approach (ii) using equivariant cohomology describe the moduli space $\mathcal{M}(n, d)$ in terms of an infinite-dimensional quotient construction; these two infinite-dimensional constructions, though very different, can be approximated by finite-dimensional quotient constructions which are very closely related to each other, and in this finite-dimensional setting the Weil conjectures provide the required link between the arithmetic and the equivariant cohomological points of view.

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